MATH104 Exam May 2006, Solutions

All questions except q.1 are standard homework examples

1. sigma, chi, ψ, β. (2 marks each.)

2.

a) $\mathbb{Z}$ contains -1, 0, and 6 but not 9.5.

b) $\{x \in \mathbb{R} \mid x > -1 \text{ and } x^2 < 20\}$ contains 0 but not -1 or 9.5 or 6.

c) $\{\frac{1}{2}n^2 - 3 \mid n \in \mathbb{Z}\}$ contains -1 (take $n = 2$) and 9.5 (take $n = 5$) but not 0 (there is no $n \in \mathbb{Z}$ with $n^2 = 6$) or 6 (there is no $n \in \mathbb{Z}$ with $n^2 = 18$).

(2 marks for a), 4 marks for b) and 6 marks for c).)

3.

a) $a \leq 1$ and $a \geq -2$ (or: $-2 \leq a \leq 1$)   (2 marks)

b) $x > y$ or $y > z$ (3 marks)

c) There exist x and y with $x > y$ and $f(x) \leq f(y)$ (3 marks)

d) There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}, x \leq y$ or $f(x) > f(y)$ (4 marks)

4.

a) f is *not injective* if there exist $x, y \in \mathbb{R}$ such that $f(x) = f(y)$ but $x \neq y$.   (2 marks)

b) $f(x)$ is injective. For let $x, y \in \mathbb{R}$ and suppose $f(x) = f(y)$. Then $8x + 12 = 8y + 12$ and rearranging gives $8x = 8y$ so $x = y$.   (4 marks)

c) $x^3 - x = 0$ for $x = 0$ and for $x = 1$. So let $x = 0, y = 1$. Then $f(x) = f(y) = 0$ but $x \neq y$. Hence f is not injective. (4 marks)

5.

For integers m and n the statement $m|n$ means that there exists an integer k with $n = km$.

a) R is not an equivalence relation. For not$(-1R-1)$ since $-1 + (-1)$ is not > 0. Hence property i) fails.

b) R is an equivalence relation. For let x, y, and z be any integers. Then

i) $x - x = 0$ and $4|0$ since $0 = 0 \times 4$, so $x\,R\,x$.

ii) If $x\,R\,y$ then $x - y = 4k$, for some $k \in \mathbb{Z}$ so $y - x = 4(-k)$, and $-k \in \mathbb{Z}$ so $y\,R\,x$.

iii) If $x\,R\,y$ and $y\,R\,z$ then $x - y = 4k$ and $y - z = 4l$ for some integers k, l so $x - z = (x - y) + (y - z) = 4(k + l)$, i.e. $x\,R\,z$.

c) R is an equivalence relation. For let x, y, and z be any elements of X. Then

i) $x|x$ (true for any integer x).

ii) If $x\,R\,y$ then $x = y$, since no element of X divides any other element. Hence $y|x$.

iii) As in ii), If $x\,R\,y$ and $y\,R\,z$ then $x = y$ and $y = z$ so $x = z$ and $x|z$.

(3 marks for a), 6 marks for b), and 5 marks for c).)

6.

a) Let m and n be integers and suppose that m and n are both odd. Then $m = 2k_1 + 1,\ n = 2k_2 + 1$ for some integers $k_1,\ k_2$. Hence $3m + 5n = 2(3k_1 + 5k_2 + 1)$, which is of the form $2k_3$ for an integer k_3 and hence even.

The converse states: Let $m, n \in \mathbb{Z}$. If $3m + 5n$ is even then m and n are odd. This is *false*: take $m = n = 2$, then $3m + 5n = 16$ which is even, but neither m nor n is odd.

b) The contrapositive of P is: Let $m, n \in \mathbb{Z}$. If m is even and n is even then $m + n$ is even.

Proof: Let $m, n \in \mathbb{Z}$ and suppose m and n are both even. Thus $m = 2k_1$ and $n = 2k_2$ for $k_1, k_2 \in \mathbb{Z}$. It follows that $m + n = 2(k_1 + k_2) = 2k$ where $k \in \mathbb{Z}$. Hence $m + n$ is even.

c) We shall prove the contrapositive:
Let $a, b \in \mathbb{R}$. If $(a + b)^2 \leq 4ab$ then $a = b$.

Proof: From $(a + b)^2 \leq 4ab$ we have $a^2 + 2ab + b^2 \leq 4ab$, so $(a - b)^2 \leq 0$. This is possible only if $a - b = 0$, that is $a = b$.

[A proof by contradiction is also of course acceptable.]

(6 marks for a), 4 marks for b), 5 marks for c))

7.

Context F and G are unfoldings. (1 mark.)

Hypothesis F is versal and F is induced from G. (1 mark.)

Conclusion G is versal. (1 mark.)

Contrapositive *Let F, G be unfoldings. If G is not versal then F is not versal or F is not induced from G.* (2 marks.)

a) Nothing. (2 marks.)

b) F is not induced from G. (3 marks.)

c) Nothing. (2 marks.)

d) F is not versal. (2 marks.)

8.

a) True. (Since $n^2 \geq 0$ we have $n^2 + 1 > 0$) (3 marks.)

b) False. ($n = 0, 1$ do not work and $n \geq 2$ makes $n^2 \geq 4$.) (3 marks.)

c) False. (Take $m = 11$ and let $n \in \mathbb{N}$; then $m + n \geq 11$ so cannot equal 10.) (3 marks.)

d) True. (Take $m = 0$.) (3 marks.)

e) True. (Take $x = \frac{1}{2}y^2$, which is > 0, hence in $\mathbb{R}^+$, and is $< y^2$.) (3 marks.)