THE UNIVERSITY
of LIVERPOOL

1. Give the names of the following (lower case) Greek letters: τ, ϵ. Write the lower case Greek letters *beta* and *phi*. [8 marks]

2. For each of the following sets S, give a function $f(n)$ such that

$$S = \{f(n) \,|\, n \in \mathbb{N}\}.$$

(Recall that $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$.)

a) $S = \{-10, -3, 4, 11, 18, 25, \ldots\}$.

b) $S = \{4, 8, 16, 32, 64 \ldots\}$.

c) $S = \mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$. [12 marks]

3. Negate each of the following statements:

a) $2x + 5y = 6$.

b) $-1 \leq a < 1$.

c) If $x = y + 2\pi$ then $f(x) = f(y)$.

d) $\forall n \in \mathbb{N}$, $\exists x \in \mathbb{R}$, $f(x) > n$. [12 marks]

4.

Definition: Let $f(x)$ be a (real-valued) function. Then $f(x)$ is *injective* if for all $x, y \in \mathbb{R}$,

$$f(x) = f(y) \implies x = y.$$

a) Write down what it means for f *not* to be injective.

b) Determine whether or not the function $f(x) = 20 - 5x$ is injective. You should justify your answer carefully, working directly from the definition.

c) Solve the equation $x^2 + x = 0$ where $x \in \mathbb{R}$. Hence show that the function $f(x) = x^2 + x$ is not injective.

[10 marks]

5. Write down carefully the meaning of the statement that $m|n$ ('m divides n'), where m and n are integers.

Definition: Let R be a relation on a set X. Then R is an *equivalence relation* if for all $x, y, z \in X$ the following three conditions hold:

i) $x\,R\,x$.

ii) If $x\,R\,y$ then $y\,R\,x$.

iii) If $x\,R\,y$ and $y\,R\,z$ then $x\,R\,z$.

Determine whether or not the following relations R on the given sets X are equivalence relations. You should justify your answers carefully, working directly from the definitions.

a) $X = \mathbb{R}$, $x\,R\,y$ if $2x \geq y$.

b) $X = \mathbb{Z}$, $x\,R\,y$ if $6|(x - y)$.

c) $X = \mathbb{Z}$, $x\,R\,y$ if $x + y$ is even. [14 marks]

6. Write proofs of the following statements. You should work from the definitions:

Definition: Let $n \in \mathbb{Z}$. Then
(i) n is *even* if there exists an integer k such that $n = 2k$.
(ii) n is *odd* if there exists an integer k such that $n = 2k + 1$.

a) Let $m, n \in \mathbb{Z}$. If m is even and n is odd then $m + n$ is odd.

b) If $n \in \mathbb{Z}$ is odd then n^2 is odd, and if $n \in \mathbb{Z}$ is even then n^2 is even.

c) It is impossible for the square of an integer n to be of the form $4m + 2$ for an integer m. (Hint: consider separately the cases where n is even and n is odd.) [15 marks]

7. Consider the following theorem. You are not expected to understand what it means.

Theorem *Let X be a T_1-space. If X is metrizable and X is separable, then X has a countable base.*

Identify the context, hypothesis, and conclusion of the theorem. State its contrapositive.

What, if anything, does the theorem tell you about a T_1-space X which is:

a) not metrizable?

b) neither metrizable nor separable?

c) metrizable and does not have a countable base?

d) has a countable base and is not separable? [14 marks]

8. Determine whether each of the following statements is true or false. Justify your answers briefly. In part e), $\mathbb{Z}^+ = \{x \in \mathbb{Z} \,|\, x > 0\}$.

a) $\exists n \in \mathbb{Z},\ n + 5 \leq 0$.

b) $\forall n \in \mathbb{Z},\ n + 5 \leq 0$.

c) $\forall x \in \mathbb{R},\ \exists y \in \mathbb{R},\ y^2 = x$.

d) $\exists x \in \mathbb{R},\ \forall y \in \mathbb{R},\ y^2 > x$.

e) $\forall n \in \mathbb{Z}^+,\ \exists x \in \mathbb{R},\ 0 < x < \frac{1}{n}$. [15 marks]