Solutions to MATH103 SEPTEMBER 2007 Examination

Note: all questions are similar to homework or examples done in class.

Section A

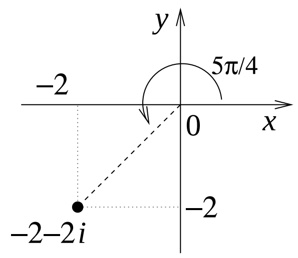
1. $z = 2 - 3i$ gives $\overline{z} = 2 + 3i$ so the required expressions is

$$\frac{1}{(1-3i)^2} = \frac{1}{-8-6i} = \frac{-8+6i}{100},$$

and the real part is $-\frac{2}{25}$ while the imaginary part is $\frac{3}{50}$.

[1 mark for $\overline{z}$, 3 marks for calculation.]

2. $|z| = \sqrt{(-2)^2 + (-2)^2} = 2\sqrt{2}$ [1 mark]
Let $\theta = \arg(z)$. Then $\tan\theta = \frac{-2}{-2} = 1$, so $\theta = \frac{\pi}{4}$ or $\frac{5\pi}{4}$. Since z is in the 3rd quadrant, $\frac{5\pi}{4}$ is correct. Thus $z = 2\sqrt{2}e^{5\pi i/4}$.



[1 mark for diagram, 1 mark for arg of z]

By de Moivre's theorem,

$$z^5 = (2\sqrt{2})^5 e^{5\times\frac{5\pi i}{4}} = 2^{15/2}e^{25\pi i/4} = 128\sqrt{2}e^{\pi i/4} = 128(1+i)\,.$$

The real part of z^5 is 128 and the imaginary part is 128. [3 marks]

3. $(5+2i)^2 = 5^2 + 2\times 5\times 2i + (2i)^2 = 25 + 20i - 4 = 21 + 20i$ [1 mark]
Thus the square roots of $21 + 20i$ are $\pm(5+2i)$. Using the quadratic formula,

$$z = \frac{-3-4i \pm\sqrt{(3+4i)^2 - 4(-7+i)}}{2} = \frac{-3-4i\pm\sqrt{9+24i-16+28-4i}}{2}$$

$$= \frac{-3-4i\pm\sqrt{21+20i}}{2} = \frac{-3-4i\pm(5+2i)}{2} = 1-i \quad\text{or}\quad -4-3i.$$

[3 marks]

4. $\mathbf{m} = (\mathbf{a}+\mathbf{b})/2$ and $\mathbf{p} = \frac{3}{5}\mathbf{c} + \frac{2}{5}\mathbf{m} = \frac{1}{5}\mathbf{a} + \frac{1}{5}\mathbf{b} + \frac{3}{5}\mathbf{c}$, so that $\overrightarrow{PA} + \overrightarrow{PB} + 3\,\overrightarrow{PC} = \mathbf{a}-\mathbf{p}+\mathbf{b}-\mathbf{p}+3(\mathbf{c}-\mathbf{p}) = (\mathbf{a}+\mathbf{b}+3\mathbf{c}) - 5\mathbf{p} = (\mathbf{a}+\mathbf{b}+3\mathbf{c}) - (\mathbf{a}+\mathbf{b}+3\mathbf{c}) = \mathbf{0}$. [4 marks]

5. (i) $\vec{AB} = (0-2, 1+1, -3-1) = (-2, 2, -4)$, $\vec{AC} = (3, 2, 1)$, [1 mark]
$\vec{AB} \times \vec{AC} = (2+8, 2-12, -4-6) = (10, -10, -10)$. Checking perpendicularity: $(10, -10, -10)\cdot(-2, 2, -4) = -20-20+40 = 0$ and $(10, -10, -10)\cdot(3, 2, 1) = 30-20-10 = 0$, as required.

[3 marks]

(ii) The area of the triangle is $\frac{1}{2}|\ \vec{AB} \times \vec{AC}\ | = \frac{1}{2}\sqrt{10^2 + (-10)^2 + (-10)^2} = 5\sqrt{3}$.

[1 mark]

Let h be the length of the perpendicular from A to BC, $\vec{BC} = (5, 0, 5)$. Then the area of the triangle is $\frac{1}{2}h|\ \vec{BC}\ | = \frac{1}{2}h\sqrt{5^2 + 0^2 + 5^2} = 5h/\sqrt{2}$. Equating this to the area found above we get $h = \sqrt{6}$. [2 marks]

(iii) A normal to the plane ABC is given by $\vec{AB} \times \vec{AC} = (10, -10, -10)$ as above. To reduce the numbers, it is better to take one tenth of it: $\mathbf{n} = (1, -1, -1)$. An equation is then

$$(\mathbf{x} - \mathbf{a}) \cdot \mathbf{n} = 0, \text{ that is } (x-2, y+1, z-1) \cdot (1, -1, -1) = 0,$$

that is $(x-2) - (y+1) - (z-1) = 0$ giving $x - y - z - 2 = 0$. We can equally well use $(\mathbf{x} - \mathbf{b}) \cdot \mathbf{n} = 0$ or $(\mathbf{x} - \mathbf{c}) \cdot \mathbf{n} = 0$; these give the same answer.

Alternatively, the plane will be $x - y - z + k = 0$ for some number k, since its normal is $(1, -1, -1)$ and substituting the coordinates of A (or B or C) in this equation gives $k = -2$. [3 marks]

6. The pairs given are of the form (x, y) so we have to solve the equations

$$\begin{aligned} p + q + r &= 8 & (1)\\ p - q + r &= -2 & (2)\\ p + 2q + 4r &= 10 & (3) \end{aligned}$$

(1)–(2) gives $2q = 10$ so $q = 5$. Then (3)–(1) gives $q + 3r = 2$ so $3r = 2 - q = -3$ giving $r = -1$. Finally (1) gives $p = 4$.

Answer: $p = 4,\ q = 5,\ r = -1$, that is $y = 4 + 5x - x^2$.
[2 marks for setting up the equations and 3 for solving them]

7. (a) $\mathbf{u} = (21, -28, -14)$, $\mathbf{v} = (-6, 8, -4)$ are *linearly independent* since the second is not a scalar multiple of the first (note that the first two components of $\mathbf{v}$ are $-2/7$ times the first two components of $\mathbf{u}$, but the third component of $\mathbf{v}$ is $2/7$ times the third component of $\mathbf{u}$). There are only two vectors so they *cannot span* $\mathbf{R}^3$. [2 marks]
(b) Putting the vectors, $\mathbf{v}$, $\mathbf{u}$, $\mathbf{w}$ as the rows of a matrix and using row reduction gives

$$\begin{pmatrix} 1 & 2 & -1 \\ 3 & -1 & 4 \\ -7 & 7 & -14 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & -1 \\ 0 & -7 & 7 \\ 0 & 21 & -21 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & -1 \\ 0 & -7 & 7 \\ 0 & 0 & 0 \end{pmatrix}$$

using $R_2 - 3R_1$, $R_3 + 7R_1$, and then $R_3 + 3R_2$. The row of zeros shows that the three vectors are *linearly dependent*. Since they are linearly dependent they *do not span* $\mathbf{R}^3$.

[3 marks]

The rows of the second matrix are $\mathbf{v}$, $\mathbf{u}-3\mathbf{v}$, $\mathbf{w}+7\mathbf{v}$ respectively, and we therefore have $\mathbf{w}+7\mathbf{v}+3(\mathbf{u}-3\mathbf{v})=\mathbf{0}$, which gives $3\mathbf{u}-2\mathbf{v}+\mathbf{w}=\mathbf{0}$. [2 marks]

8. For B we need only multiply the diagonal entries, since B is upper triangular. So $\det(B)=1\times(-2)\times 3=-6$. [1 mark]

Expanding $\det(A)$ by the top row (which contains a zero and therefore gives a short calculation) gives
$\det(A)=2(-8+5)-3(14-15)=-6+3=-3$. [3 marks]

Using rules for determinants we now get
$\det(B^2A^{-1})=(\det(B))^2/\det(A)=36/(-3)=-12$.

The matrix $B-3I$ is also upper triangular, and has diagonal entries -2, -5, 0. The determinant is the product of these, hence is 0. [2 marks]

9. (i) The eigenvalues of A are the solutions of the equation

$$0=\det(A-\lambda I)=\det\begin{pmatrix}-2-\lambda & 2\\ 2 & 1-\lambda\end{pmatrix}.$$

This gives $(-2-\lambda)(1-\lambda)-2^2=0$, that is $\lambda^2+\lambda-6=0$, which factorises as $(\lambda+3)(\lambda-2)=0$ (or solve by the quadratic formula), so that the eigenvalues are -3 and 2. [2 marks]

(ii) For $\lambda=-3$, we have $A+3I=\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}$. Hence we can take for an eigenvector $(2,-1)$ or $\mathbf{e}_1=(\frac{2}{\sqrt5},-\frac{1}{\sqrt5})$ if making it of length 1. For $\lambda=2$, we get $A-2I=\begin{pmatrix}-4 & 2\\ 2 & -1\end{pmatrix}$ and therefore for a unit eigenvector we can take $\mathbf{e}_2=(\frac{1}{\sqrt5},\frac{2}{\sqrt5})$. [5 marks]

(iii) Taking now $P=\begin{pmatrix}\frac{2}{\sqrt5} & \frac{1}{\sqrt5}\\ -\frac{1}{\sqrt5} & \frac{2}{\sqrt5}\end{pmatrix}$ and $D=\begin{pmatrix}-3 & 0\\ 0 & 2\end{pmatrix}$ we have $P^\top A\,P=D$.

[2 marks]

Section B

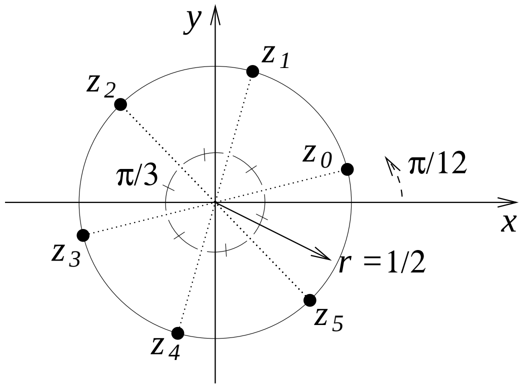
10. (i) We have $b=\frac{\sqrt2}{2}+\frac{\sqrt2}{2}i$ and $c=\frac{\sqrt3}{2}+\frac12 i$. Therefore $bc=e^{5\pi i/12}=\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12}=\frac{\sqrt6-\sqrt2}{4}+i\frac{\sqrt6+\sqrt2}{4}$, which gives $\cos\frac{5\pi}{12}=\frac{\sqrt6-\sqrt2}{4}$ and $\sin\frac{5\pi}{12}=\frac{\sqrt6+\sqrt2}{4}$ [3 marks]

(ii) $|a|=\sqrt{0^2+(1/64)^2}=1/64=2^{-6}$. Since a is purely imaginary and has positive imaginary part, its argument is $\pi/2$. Thus $a=\frac{1}{64}e^{\pi i/2}$. [2 marks]

Now write $z=re^{i\theta}$, giving $z^6=r^6e^{6i\theta}$. Equating this to $a=2^{-6}e^{\pi i/2}$ gives
$r^6=2^{-6}$, so that, r being real and >0, we have $r=1/2$,
$6\theta=\frac{\pi}{2}+2k\pi$, where we take $k=0,1,2,3,4,5$ for the distinct solutions (for $z^n=a$ we take $k=0,1,\dots,n-1$).

Hence, $\theta=\frac{\pi}{12}+\frac{k\pi}{3}$, that is, $\theta=\frac{\pi}{12},\ \frac{5\pi}{12},\ \frac{3\pi}{4},\ \frac{13\pi}{12},\ \frac{17\pi}{12}$, and $\frac{7\pi}{4}$. The solutions
$z_0=\frac12e^{\pi i/12}$, $z_1=\frac12e^{5\pi i/12}$, $z_2=\frac12e^{3\pi i/4}$, $z_3=\frac12e^{13\pi i/12}$, $z_4=\frac12e^{17\pi i/12}$, $z_5=\frac12e^{7\pi i/4}$ are indicated approximately on the diagram.

[5 marks for the solution, 3 for the diagram]



The solutions in the 1st quadrant are $z_1 = \frac{1}{2}e^{5\pi i/12} = \frac{\sqrt{6}-\sqrt{2}}{8} + i\frac{\sqrt{6}+\sqrt{2}}{8}$ and $z_0 = \frac{1}{2}e^{\pi i/12} = \frac{\sqrt{6}+\sqrt{2}}{8} + i\frac{\sqrt{6}-\sqrt{2}}{8}$ (swapping the real and imaginary parts). [2 marks]

11. (i) We have: A is invertible if and only if $\det(A) \neq 0$. Calculating $\det(A)$, evaluating by the first column gives $\det(A) = 1\cdot(3\cdot 10-(\alpha-1)\cdot(-1))-2\cdot((\alpha+2)(\alpha-1)-(-2)\cdot 3) = -2\alpha^2-\alpha+21 = (-2\alpha-7)(\alpha-3)$. Hence A is invertible if and only if $\alpha \neq -7/2,\ 3$, as required.

[4 marks for evaluating the determinant, 1 for deducing when A is invertible]

(ii) We need the inverse of $A_0 = \begin{pmatrix} 1 & 2 & -2 \\ 0 & 3 & -1 \\ -2 & -1 & 10 \end{pmatrix}$. The matrix of minors of A_0 is $\begin{pmatrix} 29 & -2 & 6 \\ 18 & 6 & 3 \\ 4 & -1 & 3 \end{pmatrix}$. The matrix of cofactors of A_0 is $\begin{pmatrix} 29 & 2 & 6 \\ -18 & 6 & -3 \\ 4 & 1 & 3 \end{pmatrix}$ and its transpose is $\begin{pmatrix} 29 & -18 & 4 \\ 2 & 6 & 1 \\ 6 & -3 & 3 \end{pmatrix}$. The determinant of A_0 is 21, using the formula found in (i). Dividing all the terms of the last matrix by the determinant we get

$$A_0^{-1} = \begin{pmatrix} \frac{29}{21} & -\frac{6}{7} & \frac{4}{21} \\ \frac{2}{21} & \frac{2}{7} & \frac{1}{21} \\ \frac{2}{7} & -\frac{1}{7} & \frac{1}{7} \end{pmatrix}$$

[6 marks]

(iii) The matrix of the left hand side of the system is A_3, with determinant zero. Using row operations on the augmented matrix of the equations we get

$$\begin{pmatrix} 1 & 5 & -2 & a \\ 0 & 3 & 2 & b \\ -2 & -1 & 10 & c \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 5 & -2 & a \\ 0 & 3 & 2 & b \\ 0 & 9 & 6 & c+2a \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -5 & -2 & a \\ 0 & 3 & 2 & b \\ 0 & 0 & 0 & c+2a-3b \end{pmatrix}$$

using $R_3 + 2R_1$ on the first step and $R_3 - 3R_2$ on the second. The required condition is the vanishing of the last term in the final third row: $2a - 3b + c = 0$. [4 marks]

12. (i)

$$\begin{aligned} x+8y+6z &= 33 \qquad (4)\\ -2x+12y+9z &= 53 \qquad (5) \end{aligned}$$

Taking (5)+2×(4) gives $28y+21z=119$, that is $4y+3z=17$ or $y=-\frac{3}{4}z+\frac{17}{4}$. Substituting in (4) gives $x=-8(-\frac{3}{4}z+\frac{17}{4})-6z+33=-1$. The parametric form of L is therefore $(-1,-\frac{3}{4}z+\frac{17}{4},z)$ or, introducing $z=4t$: $(-1,-3t+\frac{17}{4},4t)=(-1,17/4,0)+t(0,-3,4)$.
[3 marks]

(ii) $\vec{AB}=(1,6,-4)$ so the general point of L' is
$(-2,-9,10)+\lambda(1,6,-4)=(-2+\lambda,-9+6\lambda,10-4\lambda)$. [2 marks]

(iii) L' meets the plane $x-2y+z=-4$ in the point P whose parameter λ is obtained by substituting the general point as in (ii) into the equation of the plane. This gives $-2+\lambda+18-12\lambda+10-4\lambda=-4$, that is $\lambda=2$. Thus the point P of intersection is $(-2,-9,10)+2(1,6,-4)=(0,3,2)$. [3 marks]

(iv) In $\mathbf{R}^3$, the distance from point P to the line passing through point C and having direction vector $\mathbf{v}$ can be calculated using the formula

$$d=|\vec{PC}\times\mathbf{v}|/|\mathbf{v}|\,.$$

For L, we have $C=(-1,17/4,0)$ and $\mathbf{v}=(0,-3,4)$. Therefore, $|\mathbf{v}|=5$, $\vec{PC}=(-1,5/4,-2)$ and $|\vec{PC}\times\mathbf{v}|=|(-1,4,3)|=\sqrt{26}$. Hence $d=\frac{\sqrt{26}}{5}$. [3 marks]

(v) In $\mathbf{R}^3$, the distance between two non-parallel lines, one passing through point A and having direction vector $\mathbf{u}$ and the other passing through point C in direction $\mathbf{v}$, is given by the formula

$$\rho=|(\mathbf{u}\times\mathbf{v})\cdot\vec{AC}|/|\mathbf{u}\times\mathbf{v}|\,.$$

Taking L' and L for such two lines, we have:

$$\mathbf{u}=(1,6,-4)\,,\qquad \mathbf{v}=(0,-3,4)\,,\qquad \vec{AC}=(-1,17/4,0)-(-2,-9,10)=(1,53/4,-10)\,.$$

This gives

$$\mathbf{u}\times\mathbf{v}=(12,-4,-3)\,,\qquad (\mathbf{u}\times\mathbf{v})\cdot\vec{AC}=-11\,,\qquad |\mathbf{u}\times\mathbf{v}|=13\,.$$

Finally, $\rho=11/13$. [4 marks]

13. (i) Writing the vectors $\mathbf{v}_3,\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_4$ as the rows of a matrix and using row reduction gives

$$\begin{pmatrix} -1 & 2 & 0 & 3\\ 0 & 1 & -2 & 8\\ 4 & -1 & 3 & -7\\ 2 & 7 & 4 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 2 & 0 & 3\\ 0 & 1 & -2 & 8\\ 0 & 7 & 3 & 5\\ 0 & 11 & 4 & 10 \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} -1 & 2 & 0 & 3\\ 0 & 1 & -2 & 8\\ 0 & 0 & 17 & -51\\ 0 & 0 & 26 & -78 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 2 & 0 & 3\\ 0 & 1 & -2 & 8\\ 0 & 0 & 1 & -3\\ 0 & 0 & 1 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 2 & 0 & 3\\ 0 & 1 & -2 & 8\\ 0 & 0 & 1 & -3\\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where the row operations are $R_3 + 4R_1$ and $R_4 + 2R_1$, then $R_3 - 7R_2$ and $R_4 - 11R_2$, then $R_3/17$ and $R_4/26$ and finally $R_4 - R_3$. There is a row of zeros; this means that the vectors are linearly dependent. [6 marks]

(ii) The three nonzero rows of the reduced matrix in (i) have the same span as the four given vectors, and they are linearly independent because this process always results in linearly independent vectors. To simplify, we can clear the 3rd entry in the 2nd row using 3rd row, then the 2nd entry in the 1st row using the new 2nd row, and finally change the sign of the new top row. This provides $\mathbf{w}_1 = (1, 0, 0, 1)$, $\mathbf{w}_2 = (0, 1, 0, 2)$, $\mathbf{w}_3 = (0, 0, 1, -3)$ as suitable spanning vectors.

We can extend these to a basis for $\mathbf{R}^4$ by adding any row to the matrix with $\mathbf{w}_1$, $\mathbf{w}_2, \mathbf{w}_3$ as rows in such a way as to give four independent rows. The simplest thing is to add $(0, 0, 0, 1)$.

[2 marks for the independent vectors spanning S, 2 for completing to a basis]

(iii) For $\mathbf{u}_1$, we need only check whether there are scalars a, b, c such that $a(1, 0, 0, 1) + b(0, 1, 0, 2) + c(0, 0, 1, -3) = (1, 1, 1, 0)$. From the first three components we immediately read $a = b = c = 1$ which also satisfy the last component equation $a + 2b - 3c = 0$. Hence $\mathbf{u}_1 \in S$.

Similarly, for $\mathbf{u}_2$, $a(1, 0, 0, 1) + b(0, 1, 0, 2) + c(0, 0, 1, -3) = (1, -1, 1, 2)$, and we get $a = c = 1$ and $b = -1$, which this time do not satisfy the last component equation $a + 2b - 3c = 2$. Thus $\mathbf{u}_2$ is not in S. [3 marks]

(iv) From the above, $S \cap T$ is at least a line since the line in T spanned by $\mathbf{u}_1$ is in S. The most $S \cap T$ could be is the whole 2-plane T. But this is not possible as $\mathbf{u}_2$ is in T, but not in S. We conclude that $S \cap T$ is a line. [2 marks]

14. We start with the characteristic equation

$$\det(A - \lambda I) = \det \begin{pmatrix} -1-\lambda & 1 & -1 \\ 2 & 3-\lambda & -2 \\ 2 & 5 & -4-\lambda \end{pmatrix} = 0 \,.$$

Expanding along the first row and making simplifications we get:

$$(-1-\lambda)((3-\lambda)\cdot(-4-\lambda)-(-2)\cdot 5) - 1\cdot(2\cdot(-4-\lambda)-(-2)\cdot 2) - 1((2\cdot 5-(3-\lambda)\cdot 2) =$$

$$= -\lambda^3 - 2\lambda^2 + \lambda + 2 = (\lambda+2)(1-\lambda^2) = -(\lambda+2)(\lambda+1)(\lambda-1) = 0$$

Therefore, the eigenvalues are -2, -1 and 1. We can now pass to eigenvectors. [6 marks]

$\lambda = -2$. Then

$$A + 2I = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 5 & -2 \\ 2 & 5 & -2 \end{pmatrix} .$$

The cross-product of the first two rows is $(3, 0, 3)$. Diving this by 3, we take $\mathbf{v}_{-2} := (1, 0, 1)$.

$\lambda = -1$. Similarly:

$$A + I = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 4 & -2 \\ 2 & 5 & -3 \end{pmatrix} .$$

Now $R_1 \times R_2/2 = (1, -1, -1) =: v_{-1}$.

$y = 1$. Then

$$A - I = \begin{pmatrix} -2 & 1 & -1 \\ 2 & 2 & -2 \\ 2 & 5 & -5 \end{pmatrix}$$

Here $R_1 \times R_2/2 = (0, -3, -3)$. Thus we can take $\mathbf{v}_1 = (0, 1, 1)$. [7 marks]

We obtain matrix C writing the eigenvectors in columns, and we obtain matrix D writing the eigenvalues along the diagonal in the order corresponding to the eigenvectors:

$$C^{-1}AC = D\,, \qquad \text{where} \quad C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

[2 marks]