SECTION A

1. Let $z = 2 - 3i$. Find the real and imaginary parts of $\dfrac{1}{(3-\overline{z})^2}$. [4 marks]

2. Let $z = -2 - 2i$. Express z in the form $re^{i\theta}$. (As usual, $r > 0$ and θ is real.) Indicate the position of z on a diagram. Use de Moivre's theorem to find the real and imaginary parts of z^5. [6 marks]

3. Verify that $(5 + 2i)^2 = 21 + 20i$. By means of the quadratic formula, or completing the square, solve the quadratic equation

$$z^2 + (3 + 4i)z - 7 + i = 0.$$ [4 marks]

4. Let A, B, C be three points with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ respectively. Write down the position vectors
$\mathbf{m}$ of M which is the mid-point of AB;
$\mathbf{p}$ of P which is on CM, two-fifth of the distance from C to M.
Show that $\vec{PA} + \vec{PB} + 3\vec{PC}$ is the zero vector. [4 marks]

5. Let $A = (2, -1, 1)$, $B = (0, 1, -3)$ and $C = (5, 1, 2)$.

(i) Find the vectors $\vec{AB}$, $\vec{AC}$ and $\vec{AB} \times \vec{AC}$.

Verify that your vector $\vec{AB} \times \vec{AC}$ is perpendicular to the vectors $\vec{AB}$ and $\vec{AC}$, stating your method for doing this. [4 marks]

(ii) Write down the area of the triangle ABC and find the length of the perpendicular from A to the side BC. (You need not evaluate any square roots occurring.) [3 marks]

(iii) Find an equation for the plane containing the triangle ABC. [3 marks]

6. Find the values of p, q, r such that the curve $y = p + qx + rx^2$ passes through the points $(1, 8), (-1, -2)$ and $(2, 10)$. [5 marks]

7. For each set of vectors (a) and (b) decide, giving reasons, whether the vectors are linearly independent and also whether they span $\mathbf{R}^3$.

(a) $\mathbf{u} = (21, -28, -14)$, $\mathbf{v} = (-6, 8, -4)$,

(b) $\mathbf{u} = (3, -1, 4)$, $\mathbf{v} = (1, 2, -1)$, $\mathbf{w} = (-7, 7, -14)$.

If the vectors in (a) or (b) are linearly *dependent*, find a non-trivial linear combination equalling the zero vector. [7 marks]

8. Find the determinants of the matrices A and B:

$$A = \begin{pmatrix} 2 & 3 & 0 \\ -7 & 4 & 1 \\ 15 & -5 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 12 & 5 \\ 0 & -2 & 14 \\ 0 & 0 & 3 \end{pmatrix}.$$

Use the rules for determinants, which should be clearly stated, to *write down* the determinants of B^2A^{-1} and $B - 3I$, where I is the 3×3 identity matrix.

[6 marks]

9. (i) Find the eigenvalues of the matrix $A = \begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix}$. [2 marks]

(ii) For each eigenvalue, find an eigenvector of length 1. (You need not evaluate any square roots which arise.) [5 marks]

(iii) Write down an orthogonal matrix P and a diagonal matrix D such that $P^{\top}A\,P = D$. [2 marks]

SECTION B

10. (i) Express the complex numbers $b = e^{\pi i/4}$ and $c = e^{\pi i/6}$ in the cartesian form $x + iy$. Hence, considering the product bc, find $cos\,\frac{5\pi}{12}$ and $sin\,\frac{5\pi}{12}$ in the surd form.

(ii) Express the complex number $a = i/64$ in the form $|a|e^{i\alpha}$. Find all the solutions of the equation $z^6 = a$ in the form $z = re^{i\theta}$ and indicate their positions on a diagram. Making use of your result in (i) and of the diagram's symmetry, express all the solutions from the first quadrant in the exact cartesian form involving surds.

[15 marks]

11. Let

$$A = \begin{pmatrix} 1 & \alpha+2 & -2 \\ 0 & 3 & \alpha-1 \\ -2 & -1 & 10 \end{pmatrix}.$$

(i) Show that A is invertible if and only if $\alpha \neq -7/2$ and $\alpha \neq 3$. [5 marks]

(ii) Find the inverse of A when $\alpha = 0$. [6 marks]

(iii) Find a condition which a, b and c must satisfy for the system of equations

$$\begin{aligned} x + 5y - 2z &= a \\ 3y + 2z &= b \\ -2x - y + 10z &= c \end{aligned}$$

to be consistent. [4 marks]

12. Let L denote the line of intersection of the planes in $\mathbf{R}^3$ with equations

$$x + 8y + 6z = 33 \quad \text{and} \quad -2x + 12y + 9z = 53.$$

Let L' denote the line joining the points $A = (-2, -9, 10)$ and $B = (-1, -3, 6)$.

(i) Find in parametric form an expression for the general point of L. [3 marks]

(ii) Write down the vector $\vec{AB}$ and an expression for the general point of L'. [2 marks]

(iii) Determine the point P at which L' meets the plane

$$x - 2y + z = -4.$$

[3 marks]

(iv) Find the distance from the point P to the line L. [3 marks]

(v) Find the distance between the lines L and L'. [4 marks]

13. Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ in $\mathbf{R}^4$ are defined by

$$\mathbf{v}_1 = (0, 1, -2, 8),\ \mathbf{v}_2 = (4, -1, 3, -7),\ \mathbf{v}_3 = (-1, 2, 0, 3),\ \mathbf{v}_4 = (2, 7, 4, 4).$$

(i) Show that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly dependent. [6 marks]

(ii) Let S be the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$. Find linearly independent vectors with the same span S. Extend these linearly independent vectors to a basis of $\mathbf{R}^4$. [4 marks]

(iii) Decide whether the vectors $\mathbf{u}_1 = (1, 1, 1, 0)$ and $\mathbf{u}_2 = (1, -1, 1, 2)$ are in S. [3 marks]

(iv) Let T be the span of $\mathbf{u}_1$ and $\mathbf{u}_2$. What is the intersection $S \cap T$? [2 marks]

14. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} -1 & 1 & -1 \\ 2 & 3 & -2 \\ 2 & 5 & -4 \end{pmatrix}$$

(hint: one of the eigenvalues is -2). Hence write down a matrix C and a diagonal matrix D such that $C^{-1}AC = D$. [15 marks]