Solutions to MATH103 SEPTEMBER 2006 Examination

Note: all questions are similar to homework or examples done in class.

Section A

1. $z = 1 + 2i$ gives $\overline{z} - 2i = 1 - 4i$ so the required expressions is

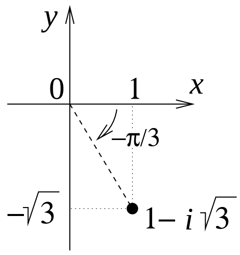
$$\frac{1}{(1-4i)^2} = \frac{1}{-15-8i} = \frac{-15+8i}{289},$$

and the real part is $-\frac{15}{289}$ while the imaginary part is $\frac{8}{289}$.

[1 mark for $\overline{z}$, 3 marks for calculation.]

2. $|z| = \sqrt{1^2 + (-\sqrt{3})^2} = 2$ [1 mark]

Let $\theta = \arg(z)$. Then $\tan\theta = \frac{-\sqrt{3}}{1} = -\sqrt{3}$, so $\theta = -\frac{\pi}{3}$ or $\frac{2\pi}{3}$. Since z is in the 4rd quadrant, $-\frac{\pi}{3}$ is correct. Thus $z = 2e^{-\pi i/3}$.



[1 mark for diagram, 2 marks for arg of z]

By de Moivre's theorem,

$$z^9 = 2^9 e^{9\times\frac{-\pi i}{3}} = 2^9 e^{-3\pi i} = 512 e^{\pi i} = -512\,.$$

The real part of z^9 is -512 and the imaginary part is 0. [2 marks]

3. $(3i-2)^2 = 2^2 - 2\times 2\times(3i) + (3i)^2 = 4 - 12i - 9 = -5 - 12i$ [1 mark]

Thus the square roots of $-5-12i$ are $\pm(3i-2)$. Using the quadratic formula,

$$z = \frac{-8+6i \pm \sqrt{(8-6i)^2 - 4(12-12i)}}{2} = -4+3i \pm \sqrt{(4-3i)^2 - (12-12i)}$$

$= -4+3i\pm\sqrt{16-24i-9-12+12i} = -4+3i\pm\sqrt{-5-12i} = -4+3i\pm(3i-2) = -6+6i$ or -2.

[3 marks]

4. $\mathbf{p} = \frac{4}{5}\mathbf{a} + \frac{1}{5}\mathbf{b}$ and $\mathbf{m} = (\mathbf{p}+\mathbf{c})/2 = \frac{4}{10}\mathbf{a} + \frac{1}{10}\mathbf{b} + \frac{5}{10}\mathbf{c}$, so that $4\,\vec{MA} + \vec{MB} + 5\,\vec{MC} = 4(\mathbf{a}-\mathbf{m}) + \mathbf{b} - \mathbf{m} + 5(\mathbf{c}-\mathbf{m}) = (4\mathbf{a}+\mathbf{b}+5\mathbf{c}) - 10\mathbf{m} = (4\mathbf{a}+\mathbf{b}+5\mathbf{c}) - (4\mathbf{a}+\mathbf{b}+5\mathbf{c}) = \mathbf{0}$.

[4 marks]

5. (i) $\vec{AB} = (-1-(-2), 2-0, 1-3) = (0,2,-2)$, $\vec{AC} = (2,4,0)$, [1 mark]

$\vec{AB}\times\vec{AC} = (8-0, -4+0, 0-4) = (8,-4,-4)$. Checking perpendicularity: $(8,-4,-4)\cdot(0,2,-2) = 0-8+8 = 0$ and $(8,-4,-4)\cdot(2,4,0) = 16-16+0 = 0$, as required.

[3 marks]

(ii) The area of the triangle is $\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}\sqrt{8^2 + (-4)^2 + (-4)^2} = 2\sqrt{6}$. [1 mark]

Let h be the length of the perpendicular from B to AC. Then the area of the triangle is $\frac{1}{2}h|\overrightarrow{AC}| = \frac{1}{2}h\sqrt{2^2 + 4^2 + 0^2} = h\sqrt{5}$. Equating this to the area found in (i) we get $h = \frac{2\sqrt{6}}{\sqrt{5}}$. [2 marks]

(iii) A normal to the plane ABC is given by $\overrightarrow{AB} \times \overrightarrow{AC} = (8, -4, -4)$ as above. To reduce the numbers, it is better to take one qutaer of it: $\mathbf{n} = (2, -1, -1)$. An equation is then

$$(\mathbf{x} - \mathbf{a}) \cdot \mathbf{n} = 0, \text{ that is } (x + 2, y, z - 3) \cdot (2, -1, -1) = 0,$$

that is $2(x + 2) - y - (z - 3) = 0$ giving $2x - y - z + 7 = 0$. We can equally well use $(\mathbf{x} - \mathbf{b}) \cdot \mathbf{n} = 0$ or $(\mathbf{x} - \mathbf{c}) \cdot \mathbf{n} = 0$; these give the same answer.

Alternatively, the plane will be $2x - y - z + k = 0$ for some number k, since its normal is $(2, -1, -1)$ and substituting the coordinates of A (or B or C) in this equation gives $k = 7$. [3 marks]

6. The pairs given are of the form (x, y) so we have to solve the equations

$$\begin{aligned} p + q + r &= 1 \qquad (1) \\ p - q + r &= 11 \qquad (2) \\ p + 2q + 4r &= 2 \qquad (3) \end{aligned}$$

(1)–(2) gives $2q = -10$ so $q = -5$. Then (3)–(1) gives $q + 3r = 1$ so $3r = 1 - q = 6$ giving $r = 2$. Finally (1) gives $p = 4$.

[2 marks for setting up the equations and 3 for solving them]

7. (a) $\mathbf{u} = (2, -8, 4)$, $\mathbf{v} = (-3, 12, -6)$ are *linearly dependent* (say, $\mathbf{u} = -\frac{2}{3}\mathbf{v}$), and $3\mathbf{u} + 2\mathbf{v}$ can be taken for a non-trivial linear combination equalling the zero vector. The two vectors span the same as just either of them, that is only a line in $\mathbf{R}^3$, not the entire $\mathbf{R}^3$. [3 marks]

(b) Putting the vectors, $\mathbf{w}$, $\mathbf{u}$, $\mathbf{v}$ as the rows of a matrix and using row reduction gives

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & 5 \\ -1 & 6 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & -3 \\ 0 & -5 & 11 \\ 0 & 8 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & -3 \\ 0 & -5 & 11 \\ 0 & 0 & \frac{93}{5} \end{pmatrix}$$

using $R_2 - 2R_1$, $R_3 + R_1$, and then $R_3 + \frac{8}{5}R_2$. The row echelon form contains no zero rows, hence the three vectors are *linearly independent*. Since they are linearly independent they *do span* $\mathbf{R}^3$. [4 marks]

8. For B we need only multiply the diagonal entries, since B is lower triangular. So $\det(B) = 2 \times (-1) \times 4 = -8$. [1 mark]

Expanding $\det(A)$ by the top row (which contains a zero and therefore gives a short calculation) gives
$\det(A) = -4(2 - 15) - 1(-6 - 6) = 52 + 12 = 64$. [3 marks]

Using rules for determinants we now get $\det(AB^{-2}) = \det(A)/(\det(B))^2 = 64/(-8)^2 = 1$. The matrix $B + 2I$ is also lower triangular, and has diagonal entries 4, 1, 6. The determinant is the product of these, hence is 24. [2 marks]

9. (i) The eigenvalues of A are the solutions of the equation

$$0 = \det(A - \lambda I) = \det \begin{pmatrix} -2-\lambda & 3 \\ 3 & 6-\lambda \end{pmatrix}.$$

This gives $(-2-\lambda)(6-\lambda) - 3^2 = 0$, that is $\lambda^2 - 4\lambda - 21 = 0$, which factorises as $(\lambda+3)(\lambda-7) = 0$ (or solve by the quadratic formula), so that the eigenvalues are -3 and 7. [2 marks]

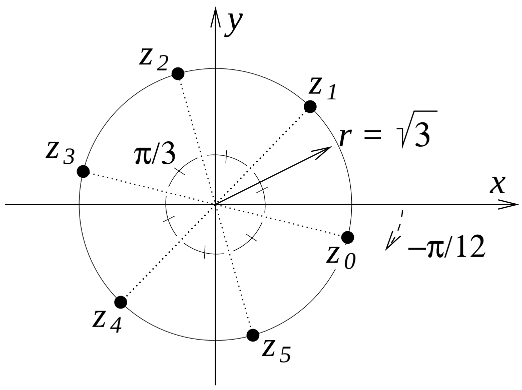
(ii) For $\lambda = -3$, we have $A+3I = \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}$. Hence we can take for an eigenvector $(3,-1)$ or $\mathbf{e}_1 = (\frac{3}{\sqrt{10}}, -\frac{1}{\sqrt{10}})$ if making it of length 1. For $\lambda = 7$, we get $A - 7I = \begin{pmatrix} -9 & 3 \\ 3 & -1 \end{pmatrix}$ and therefore for a unit eigenvector we can take $\mathbf{e}_2 = (\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}})$. [5 marks]

(iii) Taking now $P = \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{pmatrix}$ and $D = \begin{pmatrix} -3 & 0 \\ 0 & 7 \end{pmatrix}$ we have $P^\top A\, P = D$.

[2 marks]

Section B

10. $|a| = \sqrt{0^2 + (-27)^2} = 27$. Since a is purely imaginary and has negative imaginary part, its argument is $-\pi/2$. Thus $a = 27e^{-\pi i/2}$. [3 marks]



Now write $z = re^{i\theta}$, giving $z^6 = r^6 e^{6i\theta}$ and equating this to $a = 27e^{-\pi i/2}$ gives
$r^6 = 27$, so that, r being real and > 0, we have $r = \sqrt{3}$,
$6\theta = -\frac{\pi}{2} + 2k\pi$, where we take $k = 0,1,2,3,4,5$ for the distinct solutions (for $z^n = a$ we take $k = 0,1,\ldots,n-1$).
Hence, $\theta = -\frac{\pi}{12} + \frac{k\pi}{3}$, that is, $\theta = -\frac{\pi}{12}, \frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{5\pi}{4}$, and $\frac{19\pi}{12}$. The solutions $z_0 = \sqrt{3}e^{-\pi i/12}$, $z_1 = \sqrt{3}e^{\pi i/4}$, $z_2 = \sqrt{3}e^{7\pi i/12}$, $z_3 = \sqrt{3}e^{11\pi i/12}$, $z_4 = \sqrt{3}e^{5\pi i/4}$, $z_5 = \sqrt{3}e^{19\pi i/12}$ are indicated approximately on the diagram.
[6 marks for the solution, 4 for the diagram]

Two of the solutions can be easily expressed in the cartesian form: $z_1 = \frac{\sqrt{6}}{2} + \frac{\sqrt{6}}{2}i$ and $z_4 = -\frac{\sqrt{6}}{2} - \frac{\sqrt{6}}{2}i$. [2 marks]

11. (i) We have: A is invertible if and only if $\det(A) \neq 0$. Calculating $\det(A)$, evaluating by the first column gives $\det(A) = -1\cdot(-3\cdot 2 - (\alpha-2)(7-\alpha)) + 2\cdot(-3(-1) - 2(\alpha-2)) =$

$-\alpha^2 + 5\alpha + 6 = -(\alpha+1)(\alpha-6)$. Hence A is invertible if and only if $\alpha \neq 6,\ -1$, as required.

[4 marks for evaluating the determinant, 1 for deducing when A is invertible]

(ii) We need the inverse of $A_0 = \begin{pmatrix} 0 & -3 & -2 \\ 1 & 2 & -1 \\ 2 & 7 & 2 \end{pmatrix}$. The matrix of minors of A_0 is $\begin{pmatrix} 11 & 4 & 3 \\ 8 & 4 & 6 \\ 7 & 2 & 3 \end{pmatrix}$. The matrix of cofactors of A_0 is $\begin{pmatrix} 11 & -4 & 3 \\ -8 & 4 & -6 \\ 7 & -2 & 3 \end{pmatrix}$ and its transpose is $\begin{pmatrix} 11 & -8 & 7 \\ -4 & 4 & -2 \\ 3 & -6 & 3 \end{pmatrix}$. The determinant of A_0 is 6, using the formula found in (i). Dividing all the terms of the last matrix by the determinant we get

$$A_0^{-1} = \begin{pmatrix} \frac{11}{6} & -\frac{4}{3} & \frac{7}{6} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix}$$

[6 marks]

(iii) Using row operations on the augmented matrix of the equations we get

$$\begin{pmatrix} 0 & -3 & 4 & a \\ 1 & 2 & -1 & b \\ 2 & 1 & 2 & c \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & -1 & b \\ 0 & -3 & 4 & a \\ 2 & 1 & 2 & c \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & -1 & b \\ 0 & -3 & 4 & a \\ 0 & -3 & 4 & c-2b \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & -1 & b \\ 0 & -3 & 4 & a \\ 0 & 0 & 0 & c-2b-a \end{pmatrix}$$

swapping the first two rows at the start, using $R_3 - 2R_1$ after this and finishing with $R_3 - R_2$. The required condition is the vanishing of the last term in the final third row: $-a - 2b + c = 0$.

[4 marks]

12. (i)

$$\begin{aligned} x + 4y - 2z &= 3 \qquad (4) \\ 2x + 7y + 3z &= -4 \qquad (5) \end{aligned}$$

Taking $(5) - 2 \times (4)$ gives $-y + 7z = -10$, that is $y = 7z + 10$. Substituting in (4) gives $x = -4(7z+10) + 2z + 3 = -26z - 37$. The parametric form of L is therefore $(-26z - 37, 7z + 10, z)$. [It is equally valid to parametrize by x or y, but these give slightly more complicated expessions.]

[4 marks]

(ii) $\vec{AB} = (2, 0, 1)$ so the general point of L' is
$(1, -4, 2) + \lambda(2, 0, 1) = (1 + 2\lambda, -4, 2 + \lambda)$.

[3 marks]

(iii) L' meets the plane $x + y - z = -7$ in the point whose parameter λ is obtained by substituting the general point as in (ii) into the equation of the plane. This gives $1 + 2\lambda - 4 - 2 - \lambda = -7$, that is $\lambda = -2$. Thus the point of intersection is $(1, -4, 2) - 2(2, 0, 1) = (-3, -4, 0)$.

[4 marks]

(iv) In order for L to meet L' we need there to exist values of z and λ which make the general points in (i) and (ii) equal. Thus we require that the three equations $1+2\lambda = -26z-37,\quad -4 = 7z+10,\quad 2+\lambda = z$ should have a common solution for z and λ. From the second equation $z = -2$, hence $\lambda = -4$ from the third. Putting these into the first equation, we get $-7 = 15$ which cannot be true. Therefore, the lines L and L' do not meet. [4 marks]

13. (i) Writing the vectors as the rows of a matrix (starting with $\mathbf{v}_2$) and using row reduction gives

$$\begin{pmatrix} 1 & 1 & 6 & 1 \\ -2 & 0 & 9 & 4 \\ 3 & -1 & 0 & -1 \\ 1 & 3 & -3 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 6 & 1 \\ 0 & 2 & 21 & 6 \\ 0 & -4 & -18 & -4 \\ 0 & 2 & -9 & -4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 6 & 1 \\ 0 & 2 & 21 & 6 \\ 0 & 0 & 24 & 8 \\ 0 & 0 & -30 & -10 \end{pmatrix}$$

after applying R_2+2R_1 $R_3-3R_1,\ R_4-R_1$ and then $R_3+2R_1,\ R_4-R_2$. Perform now $R_3/8$ and $R_4/(-10)$, and $R_4 = R_3$ on the next step:

$$\begin{pmatrix} 1 & 1 & 6 & 1 \\ 0 & 2 & 21 & 6 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 3 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 6 & 1 \\ 0 & 2 & 21 & 6 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have ended up with a row of zeros; this means that the vectors are linearly dependent. [6 marks]

(ii) The three nonzero rows of the reduced matrix in (i) have the same span as the four given vectors, and they are linearly independent because this process always results in linearly independent vectors. To simplify and arrange more zeros (this is convenient for any further calculations), we can clear the last entries in the 1st and 2nd row using 3rd row ($R_2-6R_3,\ R_1-R_3$) and get

$$\begin{pmatrix} 1 & 1 & 3 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix},$$

where the last move was $R_1 - R_2$. Therefore, $\mathbf{w}_1 = (1,-1,0,0),\ \mathbf{w}_2 = (0,2,3,0),\ \mathbf{w}_3 = (0,0,3,1)$ are also suitable spanning vectors.

We can extend these to a basis for $\mathbf{R}^4$ by adding any row to the matrix with $\mathbf{w}_1,\ \mathbf{w}_2, \mathbf{w}_3$ as rows in such a way as to give four independent rows. The simplest thing is to add $(0,0,0,1)$.

[2 marks for the independent vectors spanning S, 3 for completing to a basis]

(iv) We need only check whether there are scalars a, b, c such that $a(1,-1,0,0)+b(0,2,3,0)+c(0,0,3,1) = (1,-3,0,1)$. From the first components we have $a = 1$, and from the second components it follows that $-1+2b = -3$ so that $b = -1$. Comparing the third components we have $-3+3c = 0$, hence $c = 1$. After this, the forth components in both sides of the equation coincide. Thus the vector $(1,-3,0,1)$ does lie in S. It is $(1,-1,0,0)-(0,2,3,0)+(0,0,3,1)$. [4 marks]

14. We start with the characteristic equation

$$\det(A - \lambda I) = \det \begin{pmatrix} -2-\lambda & 1 & 0 \\ 0 & 1-\lambda & -2 \\ -3 & 1 & 1-\lambda \end{pmatrix} = 0\,.$$

Expanding along the first row and making simplifications we get:

$$(-2-\lambda)\cdot((1-\lambda)^2 - (-2)\cdot 1) - 1\cdot(0 - (-2)\cdot(-3))$$

$$= -\lambda^3 + \lambda = 0$$

or $\lambda(\lambda-1)(\lambda+1) = 0$. Therefore, the eigenvalues are 0, 1 and -1. We can now pass to eigenvectors. [6 marks]

$\lambda = 0$. Perform row operations with the matrix A itself:

$$A = \begin{pmatrix} -2 & 1 & 0 \\ 0 & 1 & -2 \\ -3 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} -2 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} -2 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

We started with $R_3 - \frac{3}{2}R_1$ and then did $R_3 + \frac{1}{2}R_2$. The final equations are $-2x + y = 0$ and $y - 2z = 0$. As their non-trivial solution we can take $\mathbf{v}_0 = (x, y, z) = (1, 2, 1)$. In fact, any non-zero multiple of this solution would do.

$\lambda = 1$. Similarly:

$$A - I = \begin{pmatrix} -3 & 1 & 0 \\ 0 & 0 & -2 \\ -3 & 1 & 0 \end{pmatrix}$$

The top and bottom rows read $-3x + y = 0$ and the middle is $-2z = 0$. So $\mathbf{v}_1 = (1, 3, 0)$ can be taken for an eigenvector in this case.

$\lambda = -1$. Now

$$A + I = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 2 & -2 \\ -3 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 1 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 1 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

using $R_3 - 3R_1$ and $R_3 + R_2$. Now the first row reads $-x + y = 0$ and the second is $2y - 2z = 0$. So, we can set $\mathbf{v}_{-1} = (1, 1, 1)$. [7 marks]

We obtain matrix C writing the eigenvectors in columns, and we obtain matrix D writing the eigenvalues along the diagonal in the order corresponding to the eigenvectors:

$$C^{-1}AC = D\,, \qquad \text{where} \quad C = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

[2 marks]