THE UNIVERSITY *of* LIVERPOOL

SECTION A

1. Let $z = 1 + 2i$. Find the real and imaginary parts of $\dfrac{1}{(\overline{z} - 2i)^2}$. [4 marks]

2. Let $z = 1 - i\sqrt{3}$. Express z in the form $re^{i\theta}$. (As usual, $r > 0$ and θ is real.) Indicate the position of z on a diagram. Use de Moivre's theorem to find the real and imaginary parts of z^9. [6 marks]

3. Verify that $(3i - 2)^2 = -5 - 12i$. By means of the quadratic formula, or completing the square, solve the quadratic equation

$$z^2 + (8 - 6i)z + 12 - 12i = 0.$$

[4 marks]

4. Let A, B, C be three points with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ respectively. Write down the position vectors
$\mathbf{p}$ of P which is on AB, one-fifth of the distance from A to B;
$\mathbf{m}$ of M which is the mid-point of CP.
Show that $4\,\vec{MA} + \vec{MB} + 5\,\vec{MC}$ is the zero vector. [4 marks]

5. Let $A = (-2, 0, 3)$, $B = (-2, 2, 1)$ and $C = (0, 4, 3)$.

(i) Find the vectors $\vec{AB}$, $\vec{AC}$ and $\vec{AB} \times \vec{AC}$.

Verify that your vector $\vec{AB} \times \vec{AC}$ is perpendicular to the vectors $\vec{AB}$ and $\vec{AC}$, stating your method for doing this. [4 marks]

(ii) Write down the area of the triangle ABC and find the length of the perpendicular from B to the side AC. (You need not evaluate any square roots occurring.) [3 marks]

(iii) Find an equation for the plane containing the triangle ABC. [3 marks]

6. Find the values of p, q, r such that the curve $y = p + qx + rx^2$ passes through the points $(1, 1), (-1, 11)$ and $(2, 2)$. [5 marks]

7. For each set of vectors (a) and (b) decide, giving reasons, whether the vectors are linearly independent and also whether they span $\mathbf{R}^3$.

$$\text{(a)}\quad \mathbf{u} = (2, -8, 4),\ \mathbf{v} = (-3, 12, -6),$$

$$\text{(b)}\quad \mathbf{u} = (2, -1, 5),\ \mathbf{v} = (-1, 6, 4),\ \mathbf{w} = (1, 2, -3).$$

If the vectors in (a) or (b) are linearly *dependent*, find a non-trivial linear combination equalling the zero vector. [7 marks]

8. Find the determinants of the matrices A and B:

$$A = \begin{pmatrix} 0 & 4 & -1 \\ 1 & 2 & 5 \\ 3 & -6 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 & 0 \\ 8 & -1 & 0 \\ -11 & 7 & 4 \end{pmatrix}.$$

Use the rules for determinants, which should be clearly stated, to *write down* the determinants of AB^{-2} and $B + 2I$, where I is the 3×3 identity matrix.

[6 marks]

9. (i) Find the eigenvalues of the matrix $A = \begin{pmatrix} -2 & 3 \\ 3 & 6 \end{pmatrix}$. [2 marks]

(ii) For each eigenvalue, find an eigenvector of length 1. (You need not evaluate any square roots which arise.) [5 marks]

(iii) Write down an orthogonal matrix P and a diagonal matrix D such that $P^{\top} A\, P = D$. [2 marks]

SECTION B

10. Express the complex number $a = -27i$ in the form $|a|e^{i\alpha}$. Find all the solutions of the equation $z^6 = a$ in the form $z = re^{i\theta}$ and indicate their positions on a diagram. Express also two of the solutions in exact cartesian form $z = x + iy$ with no trigonometric functions involved.

[15 marks]

11. Let

$$A = \begin{pmatrix} 0 & -3 & \alpha - 2 \\ 1 & 2 & -1 \\ 2 & 7 - \alpha & 2 \end{pmatrix}.$$

(i) Show that A is invertible if and only if $\alpha \neq -1$ and $\alpha \neq 6$. [5 marks]

(ii) Find the inverse of A when $\alpha = 0$. [6 marks]

(iii) Find a condition which a, b and c must satisfy for the system of equations

$$\begin{aligned} -\ 3y + 4z &= a \\ x + 2y - z &= b \\ 2x + y + 2z &= c \end{aligned}$$

to be consistent. [4 marks]

12. Let L denote the line of intersection of the planes in $\mathbf{R}^3$ with equations

$$x + 4y - 2z = 3 \quad \text{and} \quad 2x + 7y + 3z = -4.$$

Let L' denote the line joining the points $A = (1, -4, 2)$ and $B = (3, -4, 3)$.

(i) Find in parametric form an expression for the general point of L. [4 marks]

(ii) Write down the vector $\vec{AB}$ and an expression for the general point of L'. [3 marks]

(iii) Determine the point at which L' meets the plane

$$x + y - z = -7.$$

[4 marks]

(iv) Decide whether or not L meets L'. [4 marks]

13. Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ in $\mathbf{R}^4$ are defined by

$$\mathbf{v}_1 = (-2, 0, 9, 4),\ \mathbf{v}_2 = (1, 1, 6, 1),\ \mathbf{v}_3 = (3, -1, 0, -1),\ \mathbf{v}_4 = (1, 3, -3, -3).$$

(i) Show that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly dependent. [6 marks]

(ii) Let S be the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$. Find linearly independent vectors with the same span S. Extend these linearly independent vectors to a basis of $\mathbf{R}^4$. [5 marks]

(iii) Decide whether the vector $(1, -3, 0, 1)$ lies in S. [4 marks]

14. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} -2 & 1 & 0 \\ 0 & 1 & -2 \\ -3 & 1 & 1 \end{pmatrix}$$

Hence write down a matrix C and a diagonal matrix D such that $C^{-1}AC = D$. [15 marks]