Solutions to MATH103 January 2007 Examination

Note: all questions are similar to homework or examples done in class.

Section A

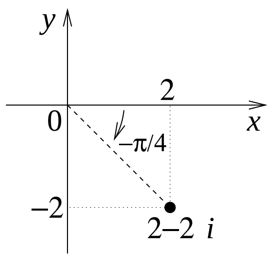
1. $z = 7 + 3i$ gives $\overline{z} = 7 - 3i$, so the required expression is

$$\frac{1}{(-1+3i)^2} = \frac{1}{-8-6i} = \frac{-8+6i}{100},$$

and the real part is $-\frac{2}{25}$ while the imaginary part is $\frac{3}{50}$.

[1 mark for $\overline{z}$, 3 marks for calculation.]

2. $|z| = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$ [1 mark]
Let $\theta = \arg(z)$. Then $\tan\theta = \frac{-2}{2} = -1$, so $\theta = -\frac{\pi}{4}$ or $\frac{3\pi}{4}$. Since z is in the 4rd quadrant, $-\frac{\pi}{4}$ is correct. Thus $z = 2\sqrt{2}e^{-\pi i/4}$.



[1 mark for diagram, 1 mark for arg of z]

By de Moivre's theorem,

$$z^5 = (2\sqrt{2})^5 e^{5\times\frac{-\pi i}{4}} = 2^{15/2}e^{-\frac{5\pi}{4}i} = 128\sqrt{2}e^{\frac{3\pi}{4}i} = 128(-1+i).$$

The real part of z^5 is -128 and the imaginary part is 128. [3 marks]

3. $(8-5i)^2 = 8^2 - 2\times 8\times 5i + (5i)^2 = 64 - 80i - 25 = 39 - 80i$ [1 mark]
Thus the square roots of $39 - 80i$ are $\pm(8-5i)$. Using the quadratic formula,

$$z = \frac{-2+3i \pm \sqrt{(2-3i)^2 - 4(-11+17i)}}{2} = \frac{-2+3i \pm \sqrt{4-12i-9+44-68i}}{2}$$

$$= \frac{-2+3i \pm \sqrt{39-80i}}{2} = \frac{-2+3i \pm (8-5i)}{2} = 3 - i \quad \text{or} \quad -5 + 4i.$$

[3 marks]

4. $\mathbf{p} = \frac{2}{5}\mathbf{a} + \frac{3}{5}\mathbf{b}$ and $\mathbf{m} = (\mathbf{p}+\mathbf{c})/2 = \frac{2}{10}\mathbf{a} + \frac{3}{10}\mathbf{b} + \frac{5}{10}\mathbf{c}$, so that $2\vec{MA} + 3\vec{MB} + 5\vec{MC} = 2(\mathbf{a}-\mathbf{m}) + 3(\mathbf{b}-\mathbf{m}) + 5(\mathbf{c}-\mathbf{m}) = (2\mathbf{a}+3\mathbf{b}+5\mathbf{c}) - 10\mathbf{m} = (2\mathbf{a}+3\mathbf{b}+5\mathbf{c}) - (2\mathbf{a}+3\mathbf{b}+5\mathbf{c}) = \mathbf{0}$.
[4 marks]

5. (i) $\vec{AB} = (0-2, 3-1, -2+2) = (-2, 2, 0)$, $\vec{AC} = (2, 2, 2)$, [1 mark]
$\vec{AB} \times \vec{AC} = (4-0, 4+0, -4-4) = (4, 4, -8)$. Checking perpendicularity: $(4,4,-8) \cdot (-2,2,0) = -8+8+0 = 0$ and $(4,4,-8)\cdot(2,2,2) = 8+8-16 = 0$, as required.
[3 marks]

(ii) The area of the triangle is $\frac{1}{2}|\vec{AB} \times \vec{AC}| = \frac{1}{2}\sqrt{4^2+4^2+(-8)^2} = 2\sqrt{6}$. [1 mark]

Let h be the length of the perpendicular from A to BC, $\vec{BC} = (4,0,2)$. Then the area of the triangle is $\frac{1}{2}h|\vec{BC}| = \frac{1}{2}h\sqrt{4^2+0^2+2^2} = h\sqrt{5}$. Equating this to the area found above we get $h = \frac{2\sqrt{6}}{\sqrt{5}} = \frac{2\sqrt{30}}{5}$. [2 marks]

(iii) A normal to the plane ABC is given by $\vec{AB} \times \vec{AC} = (4,4,-8)$ as above. To reduce the numbers, it is better to take one quater of it: $\mathbf{n} = (1,1,-2)$. An equation is then

$$(\mathbf{x}-\mathbf{a})\cdot\mathbf{n} = 0, \text{ that is } (x-2, y-1, z+2)\cdot(1,1,-2) = 0,$$

that is $(x-2)+(y-1)-2(z+2) = 0$ giving $x+y-2z-7=0$. We can equally well use $(\mathbf{x}-\mathbf{b})\cdot\mathbf{n} = 0$ or $(\mathbf{x}-\mathbf{c})\cdot\mathbf{n} = 0$; these give the same answer.

Alternatively, the plane will be $x+y-2z+k=0$ for some number k, since its normal is $(1,1,-2)$, and substituting the coordinates of A (or B or C) in this equation gives $k=-7$. [3 marks]

6. The pairs given are of the form (x, y) so we have to solve the equations

$$\begin{aligned} p+q+r &= -6 \quad (1)\\ p-2q+4r &= -15 \quad (2)\\ p+3q+9r &= -20 \quad (3) \end{aligned}$$

(1)–(2) gives $3q-3r = 9$ so $q = r+3$. Then (3)–(1) gives $2q+8r = -14$ so $10r+6 = -14$ giving $r = -2$. Hence $q = 1$. Finally (1) gives $p = -6-q-r = -5$.
Answer: $p=-5,\ q=1,\ r=-2$, that is $y = -5+x-2x^2$.
[2 marks for setting up the equations and 3 for solving them]

7. (a) $\mathbf{u} = (-8, 16, 12)$, $\mathbf{v} = (6, -12, -9)$ are *linearly dependent* (say, $\mathbf{u} = -\frac{4}{3}\mathbf{v}$), and $3\mathbf{u}+4\mathbf{v}$ can be taken for a non-trivial linear combination equalling the zero vector. The two vectors span the same as just either of them, that is only a line in $\mathbf{R}^3$, not the entire $\mathbf{R}^3$. [3 marks]

(b) Putting the vectors, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ as the rows of a matrix and using row reduction gives

$$\begin{pmatrix} 2 & -6 & 1\\ -4 & 8 & -3\\ 5 & 9 & -7 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & -6 & 1\\ 2 & -10 & 0\\ 19 & -33 & 0 \end{pmatrix}$$

using $R_2+3R_1,\ R_3+7R_1$. The determinant of the matrix obtained is $2\cdot(-33)-(-10)\cdot 19 \neq 0$, hence the three vectors are *linearly independent*. Since they are linearly independent they *do span* $\mathbf{R}^3$. [4 marks]

8. For B we need only multiply the diagonal entries, since B is lower triangular. So $\det(B) = 2 \times 1 \times (-3) = -6$. [1 mark]

Evaluating $\det(A)$ by the first column (which contains a zero and therefore gives a short calculation) gives
$\det(A) = -1(18-16) - 4(-4+3) = -2+4 = 2$. [3 marks]
Using rules for determinants we now get
$\det(B^2A^{-3}) = (\det(B))^2/(\det(A))^3 = 36/8 = 9/2$.
The matrix $B+2I$ is also lower triangular, and has diagonal entries $4,\ 3,\ -1$. The determinant is the product of these, that is -12. [2 marks]

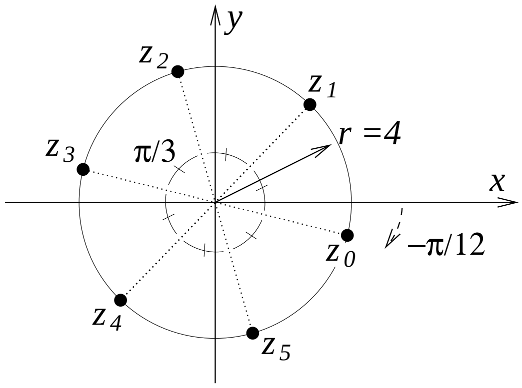
9. (i) The eigenvalues of A are the solutions of the equation $0 = \det(A-\lambda I) = \det\begin{pmatrix} 9-\lambda & 4 \\ 4 & -6-\lambda \end{pmatrix}$.
This gives $(9-\lambda)(-6-\lambda) - 4^2 = 0$, that is $\lambda^2 - 3\lambda - 70 = 0$, which factorises as $(\lambda-10)(\lambda+7) = 0$ (or solve by the quadratic formula), so that the eigenvalues are 10 and -7. [2 marks]
(ii) For $\lambda = 10$, we have $A - 10I = \begin{pmatrix} -1 & 4 \\ 4 & -16 \end{pmatrix}$. Hence we can take for an eigenvector $(4,1)$ or $\mathbf{e}_1 = (\frac{4}{\sqrt{17}}, \frac{1}{\sqrt{17}})$ if making it of length 1. For $\lambda = -7$, we get $A+7I = \begin{pmatrix} 16 & 4 \\ 4 & 1 \end{pmatrix}$ and therefore for a unit eigenvector we can take $\mathbf{e}_2 = (\frac{1}{\sqrt{17}}, -\frac{4}{\sqrt{17}})$. [5 marks]
(iii) Taking now $P = \begin{pmatrix} \frac{4}{\sqrt{17}} & \frac{1}{\sqrt{17}} \\ \frac{1}{\sqrt{17}} & -\frac{4}{\sqrt{17}} \end{pmatrix}$ and $D = \begin{pmatrix} 10 & 0 \\ 0 & -7 \end{pmatrix}$ we have $P^\top A\, P = D$.
[2 marks]

Section B

10. (i) We have $b = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ and $c = \frac{1}{2} + \frac{\sqrt{3}}{2}i$. Therefore $bc = e^{7\pi i/12} = \cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12} = \frac{\sqrt{2}-\sqrt{6}}{4} + i\frac{\sqrt{2}+\sqrt{6}}{4}$, which gives $\cos\frac{7\pi}{12} = \frac{\sqrt{2}-\sqrt{6}}{4}$ and $\sin\frac{7\pi}{12} = \frac{\sqrt{2}+\sqrt{6}}{4}$ [3 marks]
(ii) $|a| = \sqrt{0^2 + (-4^6)^2} = 4^6 = 4096$. Since a is purely imaginary and has negative imaginary part, its argument is $-\pi/2$. Thus $a = 4096e^{-\pi i/2}$. [2 marks]

Now write $z = re^{i\theta}$, giving $z^6 = r^6e^{6i\theta}$. Equating this to $a = 4^6e^{-\pi i/2}$ gives
$r^6 = 4^6$, so that, r being real and > 0, we have $r = 4$,
$6\theta = -\frac{\pi}{2} + 2k\pi$, where we take $k = 0,1,2,3,4,5$ for the distinct solutions (for $z^n = a$ we take $k = 0,1,\ldots,n-1$).
Hence, $\theta = -\frac{\pi}{12} + \frac{k\pi}{3}$, that is, $\theta = -\frac{\pi}{12}, \frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{5\pi}{4}$, and $\frac{19\pi}{12}$. The solutions
$z_0 = 4e^{-\pi i/12},\ z_1 = 4e^{\pi i/4},\ z_2 = 4e^{7\pi i/12},\ z_3 = 4e^{11\pi i/12},\ z_4 = 4e^{5\pi i/4},\ z_5 = 4e^{19\pi i/12}$ are indicated on the diagram.

[5 marks for the solution, 3 for the diagram]



The solutions in the 2nd quadrant are $z_2 = 4e^{7\pi i/12} = (\sqrt{2} - \sqrt{6}) + i(\sqrt{2} + \sqrt{6})$ and $z_3 = 4e^{11\pi i/12} = (-\sqrt{2} - \sqrt{6}) + i(\sqrt{6} - \sqrt{2})$ (swapping the moduli of the real and imaginary parts). [2 marks]

11. (i) We have: A is invertible if and only if $\det(A) \neq 0$. Calculating $\det(A)$, evaluating by the first row gives $\det(A) = -(\alpha - 2) \cdot (2 \cdot (\alpha + 1) - 1 \cdot 1) - 1 \cdot (2 \cdot (-2) - 3 \cdot 1) = -2\alpha^2 + 3\alpha + 9 = (-2\alpha - 3)(\alpha - 3)$. Hence A is invertible if and only if $\alpha \neq -3/2,\ 3$ as required.

[4 marks for evaluating the determinant, 1 for deducing when A is invertible]

(ii) We need the inverse of $A_{-1} = \begin{pmatrix} 0 & -3 & -1 \\ 2 & 3 & 1 \\ 1 & -2 & 0 \end{pmatrix}$. The matrix of minors of A_{-1} is $\begin{pmatrix} 2 & -1 & -7 \\ -2 & 1 & 3 \\ 0 & 2 & 6 \end{pmatrix}$. The matrix of cofactors of A_{-1} is $\begin{pmatrix} 2 & 1 & -7 \\ 2 & 1 & -3 \\ 0 & -2 & 6 \end{pmatrix}$ and its transpose is $\begin{pmatrix} 2 & 2 & 0 \\ 1 & 1 & -2 \\ -7 & -3 & 6 \end{pmatrix}$. The determinant of A_{-1} is 4, using the formula found in (i). Dividing all the terms of the last matrix by the determinant we get

$$A_{-1}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} \\ -\frac{7}{4} & -\frac{3}{4} & \frac{3}{2} \end{pmatrix}$$

[6 marks]

(iii) The matrix of the left hand side of the system is A_3, with determinant zero. Using row operations on the augmented matrix of the equations we get

$$\begin{pmatrix} 0 & 1 & -1 & a \\ 2 & 3 & 1 & b \\ 1 & -2 & 4 & c \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & -1 & a \\ 0 & 7 & -7 & b - 2c \\ 1 & -2 & 4 & c \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & -1 & a \\ 0 & 0 & 0 & b - 2c - 7a \\ 1 & -2 & 4 & c \end{pmatrix}$$

using $R_2 - 2R_3$ at the first step and $R_2 - 7R_1$ on the second. The required condition is the vanishing of the last term in the final second row: $7a - b + 2c = 0$. [4 marks]

12. (i)

$$\begin{aligned} x + 3y - 4z &= -16 \qquad (4) \\ 2x + 5y - 6z &= -25 \qquad (5) \end{aligned}$$

Taking $(5) - 2 \times (4)$ gives $-y + 2z = 7$, that is $y = 2z - 7$. Substituting in (4) gives $x = -3(2z-7)+4z-16 = -2z+5$. The parametric form of L is therefore $(-2z+5, 2z-7, z)$. [It is equally valid to parametrize by x or y, but these give slightly more complicated expessions.] [3 marks]

(ii) $\vec{AB} = (1, 1, -1)$ so the general point of L' is
$(7, -7, -3) + \lambda(1, 1, -1) = (7 + \lambda, -7 + \lambda, -3 - \lambda)$. [2 marks]

(iii) L' meets the plane $2x + y + 4z = -1$ in the point P whose parameter λ is obtained by substituting the general point as in (ii) into the equation of the plane. This gives $14 + 2\lambda - 7 + \lambda - 12 - 4\lambda = -1$, that is $\lambda = -4$. Thus the point P of intersection is $(7, -7, -3) - 4(1, 1, -1) = (3, -11, 1)$. [3 marks]

(iv) In $\mathbf{R}^3$, the distance from point P to the line passing through point C and having direction vector $\mathbf{v}$ can be calculated using the formula

$$d = |\, \vec{PC} \times \mathbf{v}|/|\mathbf{v}|\,.$$

For L, we have $C = (5, -7, 0)$ and $\mathbf{v} = (-2, 2, 1)$. Therefore, $|\mathbf{v}| = 3$, $\vec{PC} = (2, 4, -1)$ and $|\, \vec{PC} \times \mathbf{v}| = |(6, 0, 12)| = 6\sqrt{5}$. Hence $d = 2\sqrt{5}$. [3 marks]

(v) In $\mathbf{R}^3$, the distance between two non-parallel lines, one passing through point A and having direction vector $\mathbf{u}$ and the other passing through point C in direction $\mathbf{v}$, is given by the formula

$$\rho = |(\mathbf{u} \times \mathbf{v}) \cdot \vec{AC}\,|/|\mathbf{u} \times \mathbf{v}|\,.$$

Taking L' and L for such two lines, we have:

$$\mathbf{u} = (1, 1, -1)\,, \qquad \mathbf{v} = (-2, 2, 1)\,, \qquad \vec{AC} = (5, -7, 0) - (7, -7, -3) = (-2, 0, 3)\,.$$

This gives

$$\mathbf{u} \times \mathbf{v} = (3, 1, 4)\,, \qquad (\mathbf{u} \times \mathbf{v}) \cdot \vec{AC} = 6\,, \qquad |\mathbf{u} \times \mathbf{v}| = \sqrt{26}\,.$$

Finally, $\rho = 6/\sqrt{26}$. [4 marks]

13. (i) Writing the vectors $\mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4$ as the rows of a matrix and using row reduction gives

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 2 & 0 & 4 & 1 \\ -3 & 2 & -4 & -1 \\ 4 & -2 & 6 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 2 & 2 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & 2 & 2 & 3 \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & 1 \\ 0 & 2 & 2 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where the row operations are $R_2 - 2R_1,\ R_3 + 3R_1,\ R_4 - 4R_1$, then swapping R_2 and R_3 along with the sign change of R_2^{new}, then $R_3 - 2R_2$ and $R_4 - 2R_2$, and finally $-R_3$ and $R_4 - R_3^{new}$. There is a row of zeros; this means that the vectors are linearly dependent.

[6 marks]

(ii) The three nonzero rows of the reduced matrix in (i) have the same span as the four given vectors, and they are linearly independent because this process always results in linearly independent vectors. To simplify, we can clear the last entry in the 2nd row using 3rd row, and the 2nd entry in the top row using the new 2nd row. This provides $\mathbf{w}_1 = (1,0,2,0),\ \mathbf{w}_2 = (0,1,1,0),\ \mathbf{w}_3 = (0,0,0,1)$ as suitable spanning vectors.

We can extend these to a basis for $\mathbf{R}^4$ by adding any row to the matrix with $\mathbf{w}_1,\ \mathbf{w}_2, \mathbf{w}_3$ as rows in such a way as to give four independent rows. The simplest thing is to add $(0,0,1,0)$.

[2 marks for the independent vectors spanning S, 2 for completing to a basis]

(iii) For $\mathbf{u}_1$, we need only check whether there are scalars a, b, c such that $a(1,0,2,0) + b(0,1,1,0) + c(0,0,0,1) = (2,-3,1,-1)$. From the first components we have $a = 2$, from the second components $b = -3$ and from the last $c = -1$. The obtained values of a and b yield $4 - 3 = 1$ in the 3rd components. Hence $\mathbf{u}_1 \in S$.

Similarly for $\mathbf{u}_2$, $a(1,0,2,0) + b(0,1,1,0) + c(0,0,0,1) = (-1,1,2,0)$ implies $a = -1$, $b = 1$ and $c = 0$, which do not balance the 3rd components: $-2 + 1 \neq 2$. Thus $\mathbf{u}_2$ is not in S.

[3 marks]

(iv) From the above, $S \cap T$ is at least a line since the line in T spanned by $\mathbf{u}_1$ is in S. The most $S \cap T$ could be is the whole 2-plane T. But this is not possible as $\mathbf{u}_2$ is in T, but not in S. We conclude that $S \cap T$ is a line.

[2 marks]

14. We start with the characteristic equation

$$\det(A - \lambda I) = \det \begin{pmatrix} 4-\lambda & 1 & -4 \\ 2 & 3-\lambda & 2 \\ -3 & 3 & 5-\lambda \end{pmatrix} = 0\,.$$

Expanding along the first row and making simplifications we get:

$$(4-\lambda)\cdot((3-\lambda)(5-\lambda) - 2\cdot 3) - 1\cdot(2(5-\lambda) - 2\cdot(-3)) - 4\cdot(2\cdot 3 - (3-\lambda)\cdot(-3))$$
$$= -\lambda^3 + 12\lambda^2 - 27\lambda - 40 = 0$$

or $0 = (\lambda+1)(\lambda^2 - 13\lambda + 40) = (\lambda+1)(\lambda-5)(\lambda-8) = 0$. Therefore, the eigenvalues are -1, 5 and 8. We can now pass to eigenvectors.

[6 marks]

$\lambda = -1$. We have:

$$A + I = \begin{pmatrix} 5 & 1 & -4 \\ 2 & 4 & 2 \\ -3 & 3 & 6 \end{pmatrix}.$$

The last two rows are multiples of non-proportional vectors $(1, 2, 1)$ and $(-1, 1, 2)$, whose cross-product is $(3, -3, 3)$. For an eigenvector $\mathbf{v}_{-1}$ we take one-third of the latter:

$$\mathbf{v}_{-1} := (1, -1, 1)\,.$$

$\lambda = 5$. Then

$$A - 5I = \begin{pmatrix} -1 & 1 & -4 \\ 2 & -2 & 2 \\ -3 & 3 & 0 \end{pmatrix}.$$

The cross-product of the last two rows is $(-6, -6, 0)$. For $\mathbf{v}_5$ we can take the negative one-sixth of this:

$$\mathbf{v}_5 := (1, 1, 0)\,.$$

$\lambda = 8$. Now

$$A - 8I = \begin{pmatrix} -4 & 1 & -4 \\ 2 & -5 & 2 \\ -3 & 3 & -3 \end{pmatrix}.$$

Here $\frac{1}{3}R_3 \times R_1 = (-3, 0, 3)$, and we take one-third of it:

$$\mathbf{v}_8 := (-1, 0, 1)\,.$$

[7 marks]

We obtain matrix C writing the eigenvectors in columns, and we obtain matrix D writing the eigenvalues along the diagonal in the order corresponding to the eigenvectors:

$$C^{-1}AC = D\,, \qquad \text{where} \quad C = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

[2 marks]