SECTION A

1. Let $z = 7 + 3i$. Find the real and imaginary parts of $\dfrac{1}{(6 - \overline{z})^2}$. [4 marks]

2. Let $z = 2 - 2i$. Express z in the form $re^{i\theta}$. (As usual, $r > 0$ and θ is real.) Indicate the position of z on a diagram. Use de Moivre's theorem to find the real and imaginary parts of z^5. [6 marks]

3. Verify that $(8 - 5i)^2 = 39 - 80i$. By means of the quadratic formula, or completing the square, solve the quadratic equation

$$z^2 + (2 - 3i)z - 11 + 17i = 0.$$ [4 marks]

4. Let A, B, C be three points with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ respectively. Write down the position vectors
$\mathbf{p}$ of P which is on AB, three-fifth of the distance from A to B;
$\mathbf{m}$ of M which is the mid-point of CP.
Show that $2\vec{MA} + 3\vec{MB} + 5\vec{MC}$ is the zero vector. [4 marks]

5. Let $A = (2, 1, -2)$, $B = (0, 3, -2)$ and $C = (4, 3, 0)$.

(i) Find the vectors $\vec{AB}$, $\vec{AC}$ and $\vec{AB} \times \vec{AC}$.

Verify that your vector $\vec{AB} \times \vec{AC}$ is perpendicular to the vectors $\vec{AB}$ and $\vec{AC}$, stating your method for doing this. [4 marks]

(ii) Write down the area of the triangle ABC and find the length of the perpendicular from A to the side BC. (You need not evaluate any square roots occurring.) [3 marks]

(iii) Find an equation for the plane containing the triangle ABC. [3 marks]

6. Find the values of p, q, r such that the curve $y = p + qx + rx^2$ passes through the points $(1, -6), (-2, -15)$ and $(3, -20)$. [5 marks]

7. For each set of vectors (a) and (b) decide, giving reasons, whether the vectors are linearly independent and also whether they span $\mathbf{R}^3$.

(a) $\mathbf{u} = (-8, 16, 12)$, $\mathbf{v} = (6, -12, -9)$,

(b) $\mathbf{u} = (2, -6, 1)$, $\mathbf{v} = (-4, 8, -3)$, $\mathbf{w} = (5, 9, -7)$.

If the vectors in (a) or (b) are linearly *dependent*, find a non-trivial linear combination equalling the zero vector. [7 marks]

8. Find the determinants of the matrices A and B:

$$A = \begin{pmatrix} 0 & 2 & -1 \\ 1 & 3 & -2 \\ -4 & -16 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 & 0 \\ -15 & 1 & 0 \\ -7 & 18 & -3 \end{pmatrix}.$$

Use the rules for determinants, which should be clearly stated, to *write down* the determinants of B^2A^{-3} and $B + 2I$, where I is the 3×3 identity matrix.

[6 marks]

9. (i) Find the eigenvalues of the matrix $A = \begin{pmatrix} 9 & 4 \\ 4 & -6 \end{pmatrix}$. [2 marks]

(ii) For each eigenvalue, find an eigenvector of length 1. (You need not evaluate any square roots which arise.) [5 marks]

(iii) Write down an orthogonal matrix P and a diagonal matrix D such that $P^{\top}A\,P = D$. [2 marks]

SECTION B

10. (i) Express the complex numbers $b = e^{\pi i/4}$ and $c = e^{\pi i/3}$ in the cartesian form $x + iy$. Hence, considering the product bc, find $cos\,\frac{7\pi}{12}$ and $sin\,\frac{7\pi}{12}$ in the surd form.

(ii) Express the complex number $a = -4^6 i$ in the form $|a|e^{i\alpha}$. Find all the solutions of the equation $z^6 = a$ in the form $z = re^{i\theta}$ and indicate their positions on a diagram. Making use of your result in (i) and of the diagram's symmetry, express all the solutions from the second quadrant in the exact cartesian form involving surds.

[15 marks]

11. Let

$$A = \begin{pmatrix} 0 & \alpha - 2 & -1 \\ 2 & 3 & 1 \\ 1 & -2 & \alpha + 1 \end{pmatrix}.$$

(i) Show that A is invertible if and only if $\alpha \neq -3/2$ and $\alpha \neq 3$. [5 marks]

(ii) Find the inverse of A when $\alpha = -1$. [6 marks]

(iii) Find a condition which a, b and c must satisfy for the system of equations

$$\begin{array}{rcrcrcl} & & y & - & z & = & a \\ 2x & + & 3y & + & z & = & b \\ x & - & 2y & + & 4z & = & c \end{array}$$

to be consistent. [4 marks]

12. Let L denote the line of intersection of the planes in $\mathbf{R}^3$ with equations

$$x + 3y - 4z = -16 \quad \text{and} \quad 2x + 5y - 6z = -25.$$

Let L' denote the line joining the points $A = (7, -7, -3)$ and $B = (8, -6, -4)$.

(i) Find in parametric form an expression for the general point of L. [3 marks]

(ii) Write down the vector $\vec{AB}$ and an expression for the general point of L'. [2 marks]

(iii) Determine the point P at which L' meets the plane

$$2x + y + 4z = -1.$$

[3 marks]

(iv) Find the distance from the point P to the line L. [3 marks]

(v) Find the distance between the lines L and L'. [4 marks]

13. Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ in $\mathbf{R}^4$ are defined by

$$\mathbf{v}_1 = (2, 0, 4, 1),\ \mathbf{v}_2 = (1, -1, 1, 0),\ \mathbf{v}_3 = (-3, 2, -4, -1),\ \mathbf{v}_4 = (4, -2, 6, 3).$$

(i) Show that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly dependent. [6 marks]

(ii) Let S be the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$. Find linearly independent vectors with the same span S. Extend these linearly independent vectors to a basis of $\mathbf{R}^4$. [4 marks]

(iii) Decide whether the vectors $\mathbf{u}_1 = (2, -3, 1, -1)$ and $\mathbf{u}_2 = (-1, 1, 2, 0)$ are in S. [3 marks]

(iv) Let T be the span of $\mathbf{u}_1$ and $\mathbf{u}_2$. What is the intersection $S \cap T$? [2 marks]

14. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 4 & 1 & -4 \\ 2 & 3 & 2 \\ -3 & 3 & 5 \end{pmatrix}$$

(hint: one of the eigenvalues is -1). Hence write down a matrix C and a diagonal matrix D such that $C^{-1}AC = D$. [15 marks]