Solutions to MATH103 January 2006 Examination

Note: all questions are similar to homework or examples done in class.

Section A

1. $z = 5 - 2i$ gives $\overline{z} - 4 = 1 + 2i$ so the required expressions is

$$\frac{1}{(1+2i)^2} = \frac{1}{-3+4i} = \frac{-3-4i}{25},$$

and the real part is $-\frac{3}{25}$ while the imaginary part is $-\frac{4}{25}$.

[1 mark for $\overline{z}$, 3 marks for calculation.]

2. $|z| = \sqrt{(-\sqrt{3})^2 + (-1)^2} = 2$ [1 mark]
Let $\theta = \arg(z)$. Then $\tan\theta = \frac{-1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}$, so $\theta = \frac{\pi}{6}$ or $\frac{7\pi}{6}$. Since z is in the 3rd quadrant, $\frac{7\pi}{6}$ is correct. Thus $z = 2e^{7\pi i/6}$.

[1 mark for diagram, 2 marks for arg of z]

By de Moivre's theorem,

$$z^9 = 2^9 e^{9\times\frac{7\pi i}{6}} = 2^9 e^{\frac{21\pi}{2}i} = 512e^{\frac{\pi}{2}i} = 512i.$$

The real part of z^9 is 0 and the imaginary part is 512. [2 marks]

3. $(2+i)^2 = 2^2 + 2\times 2\times(i) + (i)^2 = 4 + 4i - 1 = 3 + 4i$ [1 mark]
Thus the square roots of $3+4i$ are $\pm(2+i)$. Using the quadratic formula,

$$z = \frac{2-7i \pm \sqrt{(2-7i)^2 - 4(-8i-12)}}{2} = \frac{2-7i \pm \sqrt{4-28i-49+32i+48}}{2}$$

$$= \frac{2-7i\pm\sqrt{3+4i}}{2} = \frac{2-7i\pm(2+i)}{2} = 2-3i \text{ or } -4i.$$

[3 marks]

4. $\mathbf{m} = (\mathbf{a}+\mathbf{b})/2$ and $\mathbf{p} = \frac{3}{4}\mathbf{c} + \frac{1}{4}\mathbf{m} = \frac{1}{8}\mathbf{a} + \frac{1}{8}\mathbf{b} + \frac{3}{4}\mathbf{c}$, so that $\vec{PA} + \vec{PB} + 6\vec{PC} = \mathbf{a}-\mathbf{p}+\mathbf{b}-\mathbf{p}+6(\mathbf{c}-\mathbf{p}) = (\mathbf{a}+\mathbf{b}+6\mathbf{c}) - 8\mathbf{p} = (\mathbf{a}+\mathbf{b}+6\mathbf{c}) - (\mathbf{a}+\mathbf{b}+6\mathbf{c}) = \mathbf{0}$. [4 marks]

5. (i) $\vec{AB} = (-1-(-1), 1-3, 2-0) = (0,-2,2)$, $\vec{AC} = (2,1,3)$, [1 mark]
$\vec{AB}\times\vec{AC} = (-6-2, 4-0, 0+4) = (-8,4,4)$. Checking perpendicularity: $(-8,4,4)\cdot(0,-2,2) = 0-8+8 = 0$ and $(-8,4,4)\cdot(2,1,3) = -16+4+12 = 0$, as required.

[3 marks]

(ii) The area of the triangle is $\frac{1}{2}|\vec{AB}\times\vec{AC}| = \frac{1}{2}\sqrt{(-8)^2+4^2+4^2} = 2\sqrt{6}$. [1 mark]

Let h be the length of the perpendicular from B to AC. Then the area of the triangle is $\frac{1}{2}h|\vec{AC}| = \frac{1}{2}h\sqrt{2^2+1^2+3^2} = \frac{1}{2}h\sqrt{14}$. Equating this to the area found in (i) we get $h = \frac{4\sqrt{6}}{\sqrt{14}} = \frac{4\sqrt{3}}{\sqrt{7}}$. [2 marks]

(iii) A normal to the plane ABC is given by $\vec{AB} \times \vec{AC} = (-8, 4, 4)$ as above. To reduce the numbers, it is better to take one qutaer of it: $\mathbf{n} = (-2, 1, 1)$. An equation is then

$$(\mathbf{x} - \mathbf{a}) \cdot \mathbf{n} = 0, \text{ that is } (x + 1, y - 3, z) \cdot (-2, 1, 1) = 0,$$

that is $-2(x + 1) + (y - 3) + z = 0$ giving $-2x + y + z - 5 = 0$. We can equally well use $(\mathbf{x} - \mathbf{b}) \cdot \mathbf{n} = 0$ or $(\mathbf{x} - \mathbf{c}) \cdot \mathbf{n} = 0$; these give the same answer.

Alternatively, the plane will be $-2x + y + z + k = 0$ for some number k, since its normal is $(-2, 1, 1)$, and substituting the coordinates of A (or B or C) in this equation gives $k = -5$. [3 marks]

6. The pairs given are of the form (x, y) so we have to solve the equations

$$\begin{aligned} p + q + r &= 2 && (1) \\ p - q + r &= -12 && (2) \\ p + 2q + 4r &= 3 && (3) \end{aligned}$$

(1)–(2) gives $2q = 14$ so $q = 7$. Then (3)–(1) gives $q + 3r = 1$ so $3r = 1 - q = -6$ giving $r = -2$. Finally (1) gives $p = -3$.

[2 marks for setting up the equations and 3 for solving them]

7. (a) $\mathbf{u} = (-3, 9, 6)$, $\mathbf{v} = (-2, 6, -4)$ are *linearly independent* since the second is not a scalar multiple of the first (note that the first two components of $\mathbf{v}$ are $2/3$ times the first two components of $\mathbf{u}$, but the third component of $\mathbf{v}$ is $-2/3$ times the third component of $\mathbf{u}$). There are only two vectors so they *cannot span* $\mathbf{R}^3$. [2 marks]

(b) Putting the vectors, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{u}$ as the rows of a matrix and using row reduction gives

$$\begin{pmatrix} -1 & 3 & 4 \\ 2 & -3 & 5 \\ -4 & 9 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 3 & 4 \\ 0 & 3 & 13 \\ 0 & -3 & -13 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 3 & 4 \\ 0 & 3 & 13 \\ 0 & 0 & 0 \end{pmatrix}$$

using $R_2 + 2R_1$, $R_3 - 4R_1$, and then $R_3 + 2R_2$. The row of zeros shows that the three vectors are *linearly dependent*. Since they are linearly dependent they *do not span* $\mathbf{R}^3$.

[3 marks]

The rows of the second matrix are $\mathbf{v}$, $\mathbf{w} + 2\mathbf{v}$, $\mathbf{u} - 4\mathbf{v}$ respectively, and we therefore have $\mathbf{w} + 2\mathbf{v} + \mathbf{u} - 4\mathbf{u} = 0$, which gives $-3\mathbf{u} + 2\mathbf{v} + \mathbf{w} = 0$. [2 marks]

8. For B we need only multiply the diagonal entries, since B is upper triangular. So $\det(B) = 3 \times 4 \times (-2) = -24$. [1 mark]

Evaluating $\det(A)$ by the third column (which contains a zero and therefore gives a short calculation) gives

$\det(A) = -1(12 - 2) - 8(-15 + 14) = -10 + 8 = -2$. [3 marks]

Using rules for determinants we now get $\det(A^{-3}B) = \det(B)/(\det(A))^3 = \frac{-24}{-8} = 3$.

The matrix $B - 4I$ is also upper triangular, and has diagonal entries -1, 0, -6. The determinant is the product of these, hence is 0.

[2 marks]

9. (i) The eigenvalues of A are the solutions of the equation $0 = \det(A - \lambda I) = \det \begin{pmatrix} 6-\lambda & -2 \\ -2 & 9-\lambda \end{pmatrix}$. This gives $(6-\lambda)(9-\lambda) - (-2)(-2) = 0$, that is $\lambda^2 - 15\lambda + 50 = 0$, which factorises as $(\lambda - 10)(\lambda - 5) = 0$ (or solve by the quadratic formula), so that the eigenvalues are 10 and 5. [2 marks]

(ii) For $\lambda = 10$, we have $A - 10I = \begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix}$. Hence we can take for an eigenvector $(1, -2)$ or $\mathbf{e}_1 = (\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}})$ if making it of length 1. For $\lambda = 5$, we get $A - 5I = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$ and therefore for a unit eigenvector we can take $\mathbf{e}_2 = (\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$. [5 marks]

(iii) Taking now $P = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$ and $D = \begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix}$ we have $P^\top A\,P = D$. [2 marks]

Section B

10. $|a| = \sqrt{0^2 + 8^2} = 8$. Since a is purely imaginary and has positive imaginary part, its argument is $\pi/2$. Thus $a = 8e^{\pi i/2}$. [3 marks]

Now write $z = re^{i\theta}$, giving $z^6 = r^6 e^{6i\theta}$ and equating this to $a = 8e^{\pi i/2}$ gives
$r^6 = 8$, so that, r being real and > 0, we have $r = \sqrt{2}$,
$6\theta = \frac{\pi}{2} + 2k\pi$, where we take $k = 0, 1, 2, 3, 4, 5$ for the distinct solutions (for $z^n = a$ we take $k = 0, 1, \ldots, n-1$).
Hence, $\theta = \frac{\pi}{12} + \frac{k\pi}{3}$, that is, $\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{17\pi}{12}$, and $\frac{7\pi}{4}$. The solutions
$z_0 = \sqrt{2}e^{\pi i/12}$, $z_1 = \sqrt{2}e^{5\pi i/12}$, $z_2 = \sqrt{2}e^{3\pi i/4}$, $z_3 = \sqrt{2}e^{13\pi i/12}$, $z_4 = \sqrt{2}e^{17\pi i/12}$, $z_5 = \sqrt{2}e^{7\pi i/4}$ are indicated approximately on the diagram.

[6 marks for the solution, 4 for the diagram]

Two of the solutions can be easily expressed in the cartesian form: $z_2 = -2 + 2i$ and $z_5 = 2 - 2i$. [2 marks]

11. (i) We have: A is invertible if and only if $\det(A) \neq 0$. Calculating $\det(A)$, evaluating by the first column gives $\det(A) = 1 \cdot (4 - (\alpha + 2)(\alpha - 7)) + 3 \cdot (-3(\alpha + 2) - 1) = -\alpha^2 - 4\alpha - 3 = -(\alpha + 1)(\alpha + 3)$. Hence A is invertible if and only if $\alpha \neq -1,\ -3$ as required.

[4 marks for evaluating the determinant, 1 for deducing when A is invertible]

(ii) We need the inverse of $A_0 = \begin{pmatrix} 1 & -3 & 1 \\ 0 & 1 & 2 \\ 3 & -7 & 4 \end{pmatrix}$. The matrix of minors of A_0 is $\begin{pmatrix} 18 & -6 & -3 \\ 5 & 1 & 2 \\ -7 & 2 & 1 \end{pmatrix}$.
The matrix of cofactors of A_0 is $\begin{pmatrix} 18 & 6 & -3 \\ -5 & 1 & -2 \\ -7 & -2 & 1 \end{pmatrix}$ and its transpose is $\begin{pmatrix} 18 & -5 & -7 \\ 6 & 1 & -2 \\ -3 & -2 & 1 \end{pmatrix}$.
The determinant of A_0 is -3, using the formula found in (i). Dividing all the terms of the last matrix by the determinant we get

$$A_0^{-1} = \begin{pmatrix} -6 & \frac{5}{3} & \frac{7}{3} \\ -2 & -\frac{1}{3} & \frac{2}{3} \\ 1 & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

[6 marks]

(iii) Using row operations on the augmented matrix of the equations we get

$$\begin{pmatrix} 1 & -3 & 1 & a \\ 0 & 1 & 1 & b \\ 3 & -8 & 4 & c \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -3 & 1 & a \\ 0 & 1 & 1 & b \\ 0 & 1 & 1 & c-3a \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -3 & 1 & a \\ 0 & 1 & 1 & b \\ 0 & 0 & 0 & c-3a-b \end{pmatrix}$$

using $R_3 - 3R_1$ at the first step and $R_3 + R_2$ on the second. The required condition is the vanishing of the last term in the final third row: $-3a + b + c = 0$. [4 marks]

12. (i)

$$x - 2y + 2z = -1 \quad (4)$$
$$3x - 7y + 4z = 5 \quad (5)$$

Taking (5) − 3 × (4) gives $-y - 2z = 8$, that is $y = -2z - 8$. Substituting in (4) gives $x = 2(-2z - 8) - 2z - 1 = -6z - 17$. The parametric form of L is therefore $(-6z-17, -2z-8, z)$. [It is equally valid to parametrize by x or y, but these give slightly more complicated expessions.] [4 marks]

(ii) $\vec{AB} = (2, 2, 4)$ so the general point of L' is
$(2, -1, -1) + \lambda(2, 2, 4) = (2 + 2\lambda, -1 + 2\lambda, -1 + 4\lambda)$. [3 marks]

(iii) L' meets the plane $x - y + z = 10$ in the point whose parameter λ is obtained by substituting the general point as in (ii) into the equation of the plane. This gives $2 + 2\lambda + 1 - 2\lambda - 1 + 4\lambda = 10$, that is $4\lambda = 8$, giving $\lambda = 2$. Thus the point of intersection is $(2, -1, -1) + 2(2, 2, 4) = (6, 3, 7)$. [3 marks]

(iv) In order for L to meet L' we need there to exist values of z and λ which make the general points in (i) and (ii) equal. Thus we require that the three equations
$2 + 2\lambda = -6z - 17$, $-1 + 2\lambda = -2z - 8$, $-1 + 4\lambda = z$ should have a common solution for z and λ. Substituting for z from the third equation into the other two gives, respectively,
$2 + 2\lambda = -6(-1 + 4\lambda) - 17$, that is $26\lambda = -13$ and
$-1 + 2\lambda = -2(-1 + 4\lambda) - 8$, that is $10\lambda = -5$.
These both give $\lambda = -1/2$ and hence $z = -1 + 4\lambda = -3$, so a common solution exists. The common point of L and L' is $(2, -1, -1) - \frac{1}{2}(2, 2, 4) = (1, -2, -3)$. [5 marks]

13. (i) Writing the vectors as the rows of a matrix and using row reduction gives

$$\begin{pmatrix} 1 & -2 & 0 & 8 \\ 2 & 1 & 15 & 5 \\ -2 & 3 & -3 & -6 \\ 1 & 0 & 6 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 & 0 & 8 \\ 0 & 5 & 15 & -11 \\ 0 & -1 & -3 & 10 \\ 0 & 2 & 6 & -6 \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} 1 & -2 & 0 & 8 \\ 0 & 1 & 3 & -11/5 \\ 0 & 0 & 0 & 39/5 \\ 0 & 0 & 0 & -8/5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 & 0 & 8 \\ 0 & 1 & 3 & -11/5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where the row operations are $R_2 - 2R_1$ $R_3 + 2R_1$, $R_4 - R_1$, then $\frac{1}{5}R_2$, $R_3 + R_2^{new}$, $R_4 - 2R_2^{new}$ and finally $\frac{5}{39}R_3$, $R_4 + \frac{8}{5}R_3^{new}$. There is a row of zeros; this means that the vectors are linearly dependent.

[6 marks]

(ii) The three nonzero rows of the reduced matrix in (i) have the same span as the four given vectors, and they are linearly independent because this process always results in linearly independent vectors. To simplify, we can clear the last entries in the 1st and 2nd row using 3rd row. This provides $\mathbf{w}_1 = (1, -2, 0, 0)$, $\mathbf{w}_2 = (0, 1, 3, 0)$, $\mathbf{w}_3 = (0, 0, 0, 1)$ as suitable spanning vectors.

We can extend these to a basis for $\mathbf{R}^4$ by adding any row to the matrix with $\mathbf{w}_1$, $\mathbf{w}_2, \mathbf{w}_3$ as rows in such a way as to give four independent rows. The simplest thing is to add $(0, 0, 1, 0)$.

[2 marks for the independent vectors spanning S, 3 for completing to a basis]

(iv) We need only check whether there are scalars a, b, c such that $a(1, -2, 0, 0) + b(0, 1, 3, 0) + c(0, 0, 0, 1) = (-4, 0, -24, 3)$. From the first components we have $a = -4$ and from the second components it follows that $8 + b = 0$ so that $b = -8$. This gives the same third components in both sides of the equation. Finally, comparing the forth components, we see that $c = 3$. Hence the vector $(-4, 0, -24, 3)$ does lie in S.

[4 marks]

14. We start with the characteristic equation

$$\det(A - \lambda I) = \det \begin{pmatrix} -2-\lambda & -5 & 3 \\ 1 & 5-\lambda & -4 \\ 1 & 7 & -6-\lambda \end{pmatrix} = 0\,.$$

Expanding along the first column and making simplifications we get:

$$(-2-\lambda)\cdot((5-\lambda)(-6-\lambda) - (-4)\cdot 7) - 1\cdot(-5(-6-\lambda) - 3\cdot 7) + 1\cdot(-5\cdot(-4) - 3(5-\lambda))$$

$$= -\lambda^3 - 3\lambda^3 - 2\lambda = 0$$

or $\lambda(\lambda+1)(\lambda+2) = 0$. Therefore, the eigenvalues are 0, -1 and -2. We can now pass to eigenvectors. [6 marks]

$\lambda = 0$. Perform row operations with the matrix A itself:

$$A = \begin{pmatrix} -2 & -5 & 3 \\ 1 & 5 & -4 \\ 1 & 7 & -6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 5 & -4 \\ 1 & 7 & -6 \\ -2 & -5 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 5 & -4 \\ 0 & 2 & -2 \\ 0 & 5 & -5 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 5 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

We started with moving the top row to the bottom of the matrix, then did $R_2 - R_1$ and $R_3 + 2R_1$, followed with $R_2/2$ and $R_3 - 5R_2^{new}$ and finished with $R_1 - 5R_2$. The final equations are $x + z = 0$ and $y - z = 0$. As their non-trivial solution we can take $\mathbf{v}_0 = (x, y, z) = (1, -1, -1)$. In fact, any non-zero multiple of this solution would do.

$\lambda = -1$. Similarly:

$$A + I = \begin{pmatrix} -1 & -5 & 3 \\ 1 & 6 & -4 \\ 1 & 7 & -5 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & -5 & 3 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

We started with $R_2 + R_1$ and $R_3 + R_1$, and then $R_3 - 2R_2$ and $R_1 + 5R_2$. From the final matrix, we see that setting $z = -1$ we obtain $x = 2$ and $y = -1$, so $\mathbf{v}_{-1} = (2, -1, -1)$ can be taken for an eigenvector in this case.

$\lambda = -2$. Now

$$A + 2I = \begin{pmatrix} 0 & -5 & 3 \\ 1 & 7 & -4 \\ 1 & 7 & -4 \end{pmatrix}$$

The 1st row reads $-5y + 3z = 0$ and we can take $y = 3$ and $z = 5$ to satisfy this equation. From the 2nd row (it is identical to the 3rd), $x = -7y + 4z = -21 + 20 = -1$. So, we set $\mathbf{v}_{-2} = (-1, 3, 5)$. [7 marks]

We obtain matrix C writing the eigenvectors in columns, and we obtain matrix D writing the eigenvalues along the diagonal in the order corresponding to the eigenvectors:

$$C^{-1}AC = D\,, \qquad \text{where} \quad C = \begin{pmatrix} 1 & 2 & -1 \\ -1 & -1 & 3 \\ -1 & -1 & 5 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

[2 marks]