SECTION A

1. Let $z = 5 - 2i$. Find the real and imaginary parts of $\dfrac{1}{(\overline{z} - 4)^2}$. [4 marks]

2. Let $z = -\sqrt{3} - i$. Express z in the form $re^{i\theta}$. (As usual, $r > 0$ and θ is real.) Indicate the position of z on a diagram. Use de Moivre's theorem to find the real and imaginary parts of z^9. [6 marks]

3. Verify that $(2 + i)^2 = 3 + 4i$. By means of the quadratic formula, or completing the square, solve the quadratic equation

$$z^2 + (7i - 2)z - 8i - 12 = 0.$$

[4 marks]

4. Let A, B, C be three points with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ respectively. Write down the position vectors

$\mathbf{m}$ of M which is the mid-point of AB;

$\mathbf{p}$ of P which is on CM, one-quater of the distance from C to M.

Show that $\vec{PA} + \vec{PB} + 6\vec{PC}$ is the zero vector. [4 marks]

5. Let $A = (-1, 3, 0)$, $B = (-1, 1, 2)$ and $C = (1, 2, 5)$.

(i) Find the vectors $\vec{AB}$, $\vec{AC}$ and $\vec{AB} \times \vec{AC}$.

Verify that your vector $\vec{AB} \times \vec{AC}$ is perpendicular to the vectors $\vec{AB}$ and $\vec{AC}$, stating your method for doing this. [4 marks]

(ii) Write down the area of the triangle ABC and find the length of the perpendicular from B to the side AC. (You need not evaluate any square roots occurring.) [3 marks]

(iii) Find an equation for the plane containing the triangle ABC. [3 marks]

6. Find the values of p, q, r such that the curve $y = p + qx + rx^2$ passes through the points $(1, 2), (-1, -12)$ and $(2, 3)$. [5 marks]

7. For each set of vectors (a) and (b) decide, giving reasons, whether the vectors are linearly independent and also whether they span $\mathbf{R}^3$.

$$\text{(a)}\quad \mathbf{u} = (-3, 9, 6),\ \mathbf{v} = (-2, 6, -4),$$

$$\text{(b)}\quad \mathbf{u} = (-4, 9, 3),\ \mathbf{v} = (-1, 3, 4),\ \mathbf{w} = (2, -3, 5).$$

If the vectors in (a) or (b) are linearly *dependent*, find a non-trivial linear combination equalling the zero vector. [7 marks]

8. Find the determinants of the matrices A and B:

$$A = \begin{pmatrix} -3 & 2 & 0 \\ -7 & 5 & 1 \\ 1 & -4 & -8 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -7 & 8 \\ 0 & 4 & -5 \\ 0 & 0 & -2 \end{pmatrix}.$$

Use the rules for determinants, which should be clearly stated, to *write down* the determinants of $A^{-3}B$ and $B - 4I$, where I is the 3×3 identity matrix.

[6 marks]

9. (i) Find the eigenvalues of the matrix $A = \begin{pmatrix} 6 & -2 \\ -2 & 9 \end{pmatrix}$. [2 marks]

(ii) For each eigenvalue, find an eigenvector of length 1. (You need not evaluate any square roots which arise.) [5 marks]

(iii) Write down an orthogonal matrix P and a diagonal matrix D such that $P^{\top} A\, P = D$. [2 marks]

SECTION B

10. Express the complex number $a = 8i$ in the form $|a|e^{i\alpha}$. Find all the solutions of the equation $z^6 = a$ in the form $z = re^{i\theta}$ and indicate their positions on a diagram. Express also two of the solutions in cartesian form $z = x + iy$ with no trigonometric functions involved.

[15 marks]

11. Let

$$A = \begin{pmatrix} 1 & -3 & 1 \\ 0 & 1 & \alpha + 2 \\ 3 & \alpha - 7 & 4 \end{pmatrix}.$$

(i) Show that A is invertible if and only if $\alpha \neq -1$ and $\alpha \neq -3$. [5 marks]

(ii) Find the inverse of A when $\alpha = 0$. [6 marks]

(iii) Find a condition which a, b and c must satisfy for the system of equations

$$\begin{array}{rcrcrcl} x & - & 3y & + & z & = & a \\ & & y & + & z & = & b \\ 3x & - & 8y & + & 4z & = & c \end{array}$$

to be consistent. [4 marks]

12. Let L denote the line of intersection of the planes in $\mathbf{R}^3$ with equations

$$x - 2y + 2z = -1 \quad \text{and} \quad 3x - 7y + 4z = 5.$$

Let L' denote the line joining the points $A = (2, -1, -1)$ and $B = (4, 1, 3)$.

(i) Find in parametric form an expression for the general point of L.
[4 marks]

(ii) Write down the vector $\vec{AB}$ and an expression for the general point of L'.
[3 marks]

(iii) Determine the point at which L' meets the plane

$$x - y + z = 10.$$

[3 marks]

(iv) Show that L meets L' and find the point of intersection. [5 marks]

13. Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ in $\mathbf{R}^4$ are defined by

$$\mathbf{v}_1 = (1, -2, 0, 8),\ \mathbf{v}_2 = (2, 1, 15, 5),\ \mathbf{v}_3 = (-2, 3, -3, -6),\ \mathbf{v}_4 = (1, 0, 6, 2).$$

(i) Show that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly dependent. [6 marks]

(ii) Let S be the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$. Find linearly independent vectors with the same span S. Extend these linearly independent vectors to a basis of $\mathbf{R}^4$.
[5 marks]

(iii) Decide whether the vector $(-4, 0, -24, 3)$ lies in S. [4 marks]

14. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} -2 & -5 & 3 \\ 1 & 5 & -4 \\ 1 & 7 & -6 \end{pmatrix}$$

Hence write down a matrix C and a diagonal matrix D such that $C^{-1}AC = D$.
[15 marks]