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SECTION A

1. Let $z = 2 + 3i$. Find the real and imaginary parts of $\overline{z} - \dfrac{1}{z}$. [4 marks]

2. Let $z = 2\sqrt{3} - 2i$. Express z in the form $re^{i\theta}$. (As usual, $r > 0$ and θ is real.) Indicate the position of z on a diagram. Use de Moivre's theorem to find the real and imaginary parts of z^6. [6 marks]

3. Verify that $(1 + 5i)^2 = -24 + 10i$. By means of the quadratic formula, or completing the square, solve the quadratic equation

$$z^2 + (i - 1)z - 3i + 6 = 0.$$ [5 marks]

4. Let A, B, C be three points with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ respectively. Write down the position vectors
$\mathbf{p}$ of P which is on BC, one-quarter of the distance from B to C;
$\mathbf{q}$ of Q which is on CA, one-quarter of the distance from C to A;
$\mathbf{r}$ of R which is on AB, one-quarter of the distance from A to B.
Show that $\mathbf{p} + \mathbf{q} + \mathbf{r} = \mathbf{a} + \mathbf{b} + \mathbf{c}$.

What can you deduce about the centroids of the triangles ABC and PQR? [4 marks]

5. Let $A = (1, 0, 2)$, $B = (-1, 3, 1)$ and $C = (0, 2, 4)$.

(i) Find the vectors $\vec{AB}$, $\vec{AC}$ and $\vec{AB} \times \vec{AC}$.

Verify that your vector $\vec{AB} \times \vec{AC}$ is perpendicular to the vectors $\vec{AB}$ and $\vec{AC}$, stating your method for doing this. [4 marks]

(ii) Write down the area of the triangle ABC and find the length of the perpendicular from B to the side AC. (You need not evaluate any square roots occurring.) [3 marks]

(iii) Find an equation for the plane containing the triangle ABC. [3 marks]

6. Find the values of p, q, r such that the curve $y = p + qx + rx^2$ passes through the points $(1, -3)$, $(2, -12)$ and $(-2, -24)$. [5 marks]

7. For each set of vectors (a) and (b) decide, giving reasons, whether the vectors are linearly independent and also whether they span $\mathbf{R}^3$.

$$\text{(a)}\quad \mathbf{u} = (2, -4, -16),\ \mathbf{v} = (-1, 2, -8),$$

$$\text{(b)}\quad \mathbf{u} = (1, 3, 5),\ \mathbf{v} = (-2, 4, 6),\ \mathbf{w} = (7, 1, 3).$$

If the vectors in (a) or (b) are linearly *dependent*, find a non-trivial linear combination equalling the zero vector. [7 marks]

8. Find the determinants of the matrices A and B:

$$A = \begin{pmatrix} 2 & 0 & 5 \\ 0 & 3 & 6 \\ 0 & 0 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 5 \\ 3 & 1 & -2 \\ 4 & 4 & 0 \end{pmatrix}.$$

Use the rules for determinants, which should be clearly stated, to *write down* the determinants of AB^{-1} and $A + 5I$, where I is the 3×3 identity matrix. [6 marks]

9. Find the eigenvalues of the matrix $A = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix}$. [3 marks]

10. Let

$$B = \begin{pmatrix} 6 & 1 & 0 \\ -2 & 1 & 4 \\ 0 & 2 & -3 \end{pmatrix}.$$

Find a nonzero vector $\mathbf{v} = (x, y, z)^\top$ satisfying $(B - 3I)\mathbf{v} = \mathbf{0}$, where I is the 3×3 identity matrix. Which real number λ is therefore an eigenvalue of B? Write down a corresponding *unit length* eigenvector. (You need not evaluate any square roots which arise.) [5 marks]

SECTION B

11. Express the complex number $a = -8\sqrt{3} + 8i$ in the form $|a|e^{i\alpha}$. Find all the solutions of the equation $z^4 = a$ in the form $z = re^{i\theta}$ and indicate their positions on a diagram. For *one* of the solutions, express it also in cartesian form $z = x + iy$ correct to 2 decimal places.

Write down, with a brief explanation, *one* solution of the equation $w^4 = \overline{a} = -8\sqrt{3} - 8i$, in cartesian form $x + iy$ correct to two decimal places. [15 marks]

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12. Let

$$A = \begin{pmatrix} 2 & -1 & 2\alpha - 1 \\ 0 & \alpha + 1 & 2 \\ 3 & -2 & 1 \end{pmatrix}.$$

(i) Show that A is invertible if and only if $\alpha \neq 1$ and $\alpha \neq -\frac{7}{6}$. [5 marks]

(ii) Find the inverse of A when $\alpha = -1$. [5 marks]

(iii) Find a condition which a, b and c must satisfy for the system of equations

$$\begin{array}{rcrcrcl} 2x & - & y & + & z & = & a \\ & & 2y & + & 2z & = & b \\ 3x & - & 2y & + & z & = & c \end{array}$$

to be consistent. [5 marks]

13. Let L denote the line of intersection of the planes in $\mathbf{R}^3$ with equations

$$x - 3y + 2z = -2 \quad \text{and} \quad 2x - 5y - z = 4.$$

Let L' denote the line joining the points $A = (-2, 2, 4)$ and $B = (-1, 4, 7)$.

(i) Find in parametric form an expression for the general point of L. [4 marks]

(ii) Write down the vector $\vec{AB}$ and an expression for the general point of L'. [3 marks]

(iii) Determine the point at which L' meets the plane

$$x + y + z = 16.$$

[3 marks]

(iv) Show that L meets L' and find the point of intersection. [5 marks]

14. Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ in $\mathbf{R}^4$ are defined by

$$\mathbf{v}_1 = (1, 1, 2, -3), \ \mathbf{v}_2 = (3, 5, 10, -4), \ \mathbf{v}_3 = (-2, 2, 1, 18), \ \mathbf{v}_4 = (1, -1, 4, -12).$$

(i) Show that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly dependent. [5 marks]

(ii) Let S be the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$. Find linearly independent vectors with the same span S. Extend these linearly independent vectors to a basis of $\mathbf{R}^4$. [5 marks]

(iii) Decide whether the vector $(-2, 4, 2, 23)$ lies in S. [5 marks]