Solutions to MATH103 January 2005 Examination

Note: all questions are similar to homework or examples done in class.

SECTION A

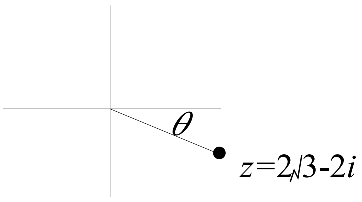
1. $z = 2 + 3i$ gives $\overline{z} = 2 - 3i$ so the required expressions is

$$2 - 3i - \frac{1}{2+3i} \times \frac{2-3i}{2-3i} = 2 - 3i - \frac{2-3i}{2^2+3^2} = 2 - 3i - \frac{2-3i}{13},$$

and the real part is $2 - \frac{2}{13} = \frac{24}{13}$ while the imaginary part is $-3 + \frac{3}{13} = \frac{-36}{13}$.

[1 mark for $\overline{z}$, 3 marks for calculation.]

2. $|z| = \sqrt{(2\sqrt{3})^2 + (-2)^2} = \sqrt{12+4} = 4$ [1 mark]
Let $\theta = \arg(z)$. Then $\tan\theta = \frac{-2}{2\sqrt{3}} = -\frac{1}{\sqrt{3}}$, so $\theta = -\frac{\pi}{6}$ or $\frac{5\pi}{6}$. Clearly from the diagram, $-\frac{\pi}{6}$ is correct. Thus $z = 4e^{-\pi i/6}$.



[1 mark for diagram, 2 marks for arg of z]

By de Moivre's theorem,

$$z^6 = 4^6 e^{6\times\frac{-\pi i}{6}} = 4^6 e^{-\pi i} = 4^6(-1) = -4^6,$$

since $e^{-\pi i} = e^{\pi i} = -1$. The real part of z^4 is -4^6 and the imaginary part is 0. [2 marks]

3. $(1+5i)^2 = 1^2 + 2\times 1 \times (5i) + (5i)^2 = 1 + 10i - 25 = -24 + 10i$ [1 mark]
Thus the square roots of $-24 + 10i$ are $\pm(1+5i)$. Using the quadratic formula,

$$z = \frac{1 - i \pm \sqrt{(i-1)^2 - 4(-3i+6)}}{2} = \frac{1 - i \pm \sqrt{-1 - 2i + 1 + 12i - 24}}{2}$$

$$= \frac{1 - i \pm \sqrt{-24 + 10i}}{2} = \frac{1 - i \pm (1+5i)}{2} = 1 + 2i \text{ or } -3i.$$

[4 marks]

4. $\mathbf{p} = \frac{3}{4}\mathbf{b} + \frac{1}{4}\mathbf{c}$, $\mathbf{q} = \frac{3}{4}\mathbf{c} + \frac{1}{4}\mathbf{a}$, $\mathbf{r} = \frac{3}{4}\mathbf{a} + \frac{1}{4}\mathbf{b}$, so that, adding these equations, $\mathbf{p} + \mathbf{q} + \mathbf{r} = \mathbf{a} + \mathbf{b} + \mathbf{c}$. [3 marks]

The centroid of a triangle has position vector one-third the sum of the position vectors of the three vertices so that ABC and PQR have the same centroid. [1 mark]

5. (i) $\vec{AB} = (-1-1, 3-0, 1-2) = (-2, 3, -1)$, $\vec{AC} = (-1, 2, 2)$, [1 mark]
$\vec{AB} \times \vec{AC} = (3(2) - (-1)(2), (-1)(-1) - (-2)(2), (-2)(2) - 3(-1)) = (8, 5, -1)$. Checking perpendicularity: $(8,5,-1)\cdot(-2,3,-1) = -16 + 15 + 1 = 0$ and $(8,5,-1)\cdot(-1,2,2) = -8 + 10 - 2 = 0$, as required. [2 marks]

The area of the triangle is $\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}\sqrt{(8)^2 + 5^2 + (-1)^2} = \frac{1}{2}\sqrt{90}$. [1 mark]

(ii) Let h be the length of the perpendicular from B to AC. Then the area of the triangle is $\frac{1}{2}h|\overrightarrow{AC}| = \frac{1}{2}h\sqrt{(-1)^2 + 2^2 + 2^2} = \frac{1}{2}h\sqrt{9} = \frac{3}{2}h$. Equating this to the area found in (i) we get $h = \frac{2}{3} \times \frac{1}{2}\sqrt{90} = \frac{1}{3}\sqrt{90}$. In fact this simplifies to $\sqrt{10}$ since $\sqrt{90} = \sqrt{9 \times 10} = \sqrt{9} \times \sqrt{10} = 3\sqrt{10}$. [3 marks]

(iii) A normal to the plane ABC is given by $\overrightarrow{AB} \times \overrightarrow{AC} = (8, 5, -1)$ as above. An equation is then

$$(\mathbf{x} - \mathbf{a}) \cdot \mathbf{n} = 0, \text{ that is } (x - 1, y, z - 2) \cdot (8, 5, -1) = 0,$$

that is $8(x - 1) + 5y + (-1)(z - 2) = 0$ giving $8x + 5y - z = 6$. We can equally well use $(\mathbf{x} - \mathbf{b}) \cdot \mathbf{n} = 0$ or $(\mathbf{x} - \mathbf{c}) \cdot \mathbf{n} = 0$; these give the same answer.

Alternatively, the plane will be $8 + 5y - z = k$ for some number k, since its normal is $(8, 5, -1)$, and substituting the coordinates of A (or B or C) in this equation gives $k = 6$. [3 marks]

6. The pairs given are of the form (x, y) so we have to solve the equations

$$\begin{aligned} p + q + r &= -3 & (1) \\ p + 2q + 4r &= -12 & (2) \\ p - 2q + 4r &= -24 & (3) \end{aligned}$$

(2)–(3) gives $4q = 12$ so $q = 3$. Then (2)–(1) gives $q + 3r = -9$ so $3r = -9 - q = -12$ giving $r = -4$. Finally (1) gives $p = -2$.

[2 marks for setting up the equations and 3 for solving them]

7. (a) $\mathbf{v} = (2, -4, -16)$, $\mathbf{w} = (-1, 2, -8)$ are *linearly independent* since the second is not a scalar multiple of the first (note that the first two components of $\mathbf{v}$ are –2 times the first two components of $\mathbf{w}$, but the third component of $\mathbf{v}$ is +2 times the third component of $\mathbf{w}$). There are only two vectors so they *cannot span* $\mathbf{R}^3$. [2 marks]

(b) Putting the vectors, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ as the rows of a matrix and using row reduction gives

$$\begin{pmatrix} 1 & 3 & 5 \\ -2 & 4 & 6 \\ 7 & 1 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & 5 \\ 0 & 10 & 16 \\ 0 & -20 & -32 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & 5 \\ 0 & 10 & 16 \\ 0 & 0 & 0 \end{pmatrix}$$

using $R_2 + 2R_1$, $R_3 - 7R_1$, and then $R_3 + 2R_2$. The row of zeros shows that the three vectors are *linearly dependent.* Since they are linearly dependent they *do not span* $\mathbf{R}^3$. [3 marks]

The rows of the second matrix are $\mathbf{u}$, $\mathbf{v} + 2\mathbf{u}$, $\mathbf{w} - 7\mathbf{u}$ respectively, and we therefore have $\mathbf{w} - 7\mathbf{u} = -2(\mathbf{v} + 2\mathbf{u})$, which gives $3\mathbf{u} - 2\mathbf{v} - \mathbf{w} = 0$. [2 marks]

8. For A we need only multiply the diagonal entries, since A is upper triangular. So $\det(A) = 2 \times 3 \times -4 = -24$. [1 mark]

Evaluating $\det(B)$ by the third column (which contains a zero and therefore gives a short calculation) gives

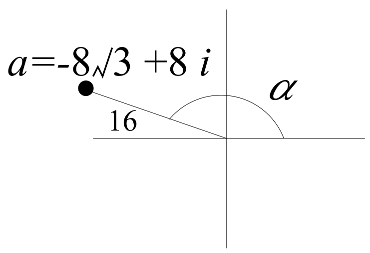
$\det(B) = 5(3\times 4 - 1\times 4) + 2(1\times 4 - 2\times 4) = 40 - 8 = 32$. (Note the $+2$ in front of the second bracket is because the entry -2 in B is in a 'minus' position.) [3 marks]
Using rules for determinants we now get $\det(AB^{-1}) = \det(A)/\det(B) = \frac{-24}{32} = -\frac{3}{4}$.
The matrix $A + 5I$ is also upper triangular, and has diagonal entries 7, 8, 1. The determinant is the product of these, hence is 56. [2 marks]

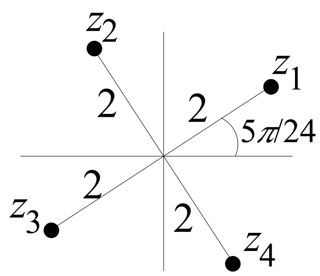
9. The eigenvalues of A are the solutions of the equation $0 = \det(A - \lambda I) = \begin{pmatrix} 1-\lambda & 3 \\ 5 & 3-\lambda \end{pmatrix}$.
This gives $(1-\lambda)(3-\lambda) - (3)(5) = 0$, that is $\lambda^2 - 4\lambda - 12 = 0$, which factorises as $(\lambda - 6)(\lambda + 2) = 0$ (or solve by the quadratic formula), so that the eigenvalues are 6 and -2. [3 marks]

10. $B - 3I = \begin{pmatrix} 3 & 1 & 0 \\ -2 & -2 & 4 \\ 0 & 2 & -6 \end{pmatrix}$ and we want to solve the equations in x, y, z with this coefficient matrix. The first equation (row) gives $3x + y = 0$ and the third equation gives $2y - 6z = 0$, that is $y = 3z$. The solution $(-z, 3z, z)$ then satisfies all three equations. A nonzero solution is $(-1, 3, 1)$. Having found a nonzero solution it follows that 3 is an eigenvalue of B. (More generally, if there is a nonzero solution of $\det(B - \lambda I) = 0$, then λ is an eigenvalue.) [3 marks]
A unit eigenvector is $\dfrac{(-1,3,1)}{\sqrt{(-1)^2 + 3^2 + 1^2}} = \dfrac{(-1,3,1)}{\sqrt{11}}$. [The only other unit eigenvector is minus this vector.] [2 marks]

Section B

11. $|a| = \sqrt{(-8\sqrt{3})^2 + 8^2} = \sqrt{192 + 64} = 16$. (Alternatively, $a = 8(-\sqrt{3} + i)$ so $|a| = 8\sqrt{3+1} = 16$.) Next, $\tan\alpha = \frac{8}{-8\sqrt{3}} = -\frac{1}{\sqrt{3}}$ and one solution for this is $\alpha = -\frac{\pi}{6}$, the other being $-\frac{\pi}{6} + \pi = \frac{5\pi}{6}$, From the left-hand diagram, the latter is clearly the correct value. Thus $a = 16e^{5\pi i/6}$. [4 marks]





The angles between the z's are all $\pi/2$

Now write $z = re^{i\theta}$, giving $z^4 = r^4 e^{4i\theta}$ and equating this to $a = 16e^{5\pi i/6}$ gives
$r^4 = 16$, so that, r being real and > 0, we have $r = 2$,
$4\theta = \frac{5\pi}{6} + 2k\pi$, where we take $k = 0, 1, 2, 3$ for the distinct solutions (for $z^n = a$ we take $k = 0, 1, \ldots, n-1$).
Hence, $\theta = \frac{5\pi}{24} + \frac{k\pi}{4}$, that is, $\theta = \frac{5\pi}{24}, \frac{17\pi}{24}, \frac{29\pi}{24}$ and $\frac{41\pi}{24}$ (the last two can be replaced by $\frac{17\pi}{24} - 2\pi = -\frac{19\pi}{24}$ and $\frac{41\pi}{24} - 2\pi = -\frac{7\pi}{24}$ if the principal value, between $-\pi$ and π, is wanted).

The solutions
$z_1 = 2e^{5\pi i/24}$, $z_2 = 2e^{17\pi i/24}$, $z_3 = 2e^{29\pi i/24} = 2e^{41\pi i/24}$ are indicated approximately on the right-hand diagram above. [4 marks for the solution, 3 for the diagram]

Using a calculator, the solution $z_1 = 2(\cos(\frac{5\pi}{24}) + i\sin(\frac{5\pi}{24})) = 1.59 + 1.22i$ to two decimal places. [Writing this as $x + iy$ the other three solutions are $-y + ix, -x - iy$ and $y - ix$.] [2 marks]

The solutions of $w^3 = \overline{a}$ are the conjugates of the solutions of $z^3 = a$, since $z^3 = a$ gives $\overline{a} = \overline{z^3} = \overline{z}^3$. Therefore one solution is $w = 1.59 - 1.22i$. [2 marks]

12. (i) We have: A is invertible if and only if $\det(A) \neq 0$. Calculating $\det(A)$, evaluating by the first column gives $\det(A) = 2(\alpha+1+4)+3(-2-(2\alpha^2+\alpha-1)) = -(6\alpha^2+\alpha-7) = -(6\alpha+7)(\alpha-1)$. Hence A is invertible if and only if $\alpha \neq -\frac{7}{6}$, 1 as required.

[4 marks for evaluating the determinant, 1 for deducing when A is invertible]

(ii) We need the inverse of $A_0 = \begin{pmatrix} 2 & -1 & -3 \\ 0 & 0 & 2 \\ 3 & -2 & 1 \end{pmatrix}$. The matrix of minors of A_0 is $\begin{pmatrix} 4 & -6 & 0 \\ -7 & 11 & -1 \\ -2 & 4 & 0 \end{pmatrix}$ The matrix of cofactors of A_0 is $\begin{pmatrix} 4 & 6 & 0 \\ 7 & 11 & 1 \\ -2 & -4 & 0 \end{pmatrix}$ and the determinant of A_0 is 2, using the formula found in (i). Transposing the matrix of cofactors and dividing by the determinant we get

$$A_0^{-1} = \begin{pmatrix} 2 & \frac{7}{2} & -1 \\ 3 & \frac{11}{2} & -2 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$ [5 marks]

(iii) Using row operations on the augmented matrix of the equations we get

$$\begin{pmatrix} 2 & -1 & 1 & a \\ 0 & 2 & 2 & b \\ 3 & -2 & 1 & c \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & -1 & 1 & a \\ 0 & 2 & 2 & b \\ 0 & -1 & -1 & 2c-3a \end{pmatrix}$$

using $2R_3 - 3R_1$. Now a further operation $2R_3 + R_2$ gives a third row $(0, 0, 0, 4c - 6a + b)$, so that the required condition is $-6a + b + 4c = 0$. [5 marks]

13. (i)

$$x - 3y + 2z = -2 \qquad (4)$$
$$2x - 5y - z = 4 \qquad (5)$$

Taking (5) $- 2 \times$ (4) gives $y - 5z = 8$, that is $y = 5z + 8$. Substituting in (4) gives $x = 3(5z+8)-2z-2 = 13z+22$. The parametric form of L is therefore $(13z+22, 5z+8, z)$. [It is equally valid to parametrize by x or y, but these give slightly more complicated expessions.] [4 marks]

(ii) $\vec{AB} = (1, 2, 3)$ so the general point of L' is
$(-2, 2, 4) + \lambda(1, 2, 3) = (-2+\lambda, 2+2\lambda, 4+3\lambda)$. [3 marks]

(iii) L' meets the plane $x + y + z = 16$ in the point whose parameter λ is obtained by substituting the general point as in (ii) into the equation of the plane. This gives

$-2+\lambda+2+2\lambda+4+3\lambda=16$, that is $6\lambda+4=16$, giving $\lambda=2$. Thus the point of intersection is $(-2,2,4)+2(1,2,3)=(0,6,10)$. [3 marks]

(iv) In order for L to meet L' we need there to exist values of z and λ which make the general points in (i) and (ii) equal. Thus we require that the three equations $-2+\lambda=13z+22,\quad 2+2\lambda=5z+8,\quad 4+3\lambda=z$ should have a common solution for y and λ. Substituting for z from the third equation into the other two gives, respectively,
$-2+\lambda=13(4+3\lambda)+22$, that is $-76=38\lambda$ and
$2+2\lambda=5(4+3\lambda)+8$, that is $-26=13\lambda$.
These both give $\lambda=-2$ and hence $z=4+3\lambda=-2$, so a common solution exists. The common point of L and L' is $(-2,2,4)-2(1,2,3)=(-4,-2,-2)$. [5 marks]

14. (i) Writing the vectors as the rows of a matrix and using row reduction gives

$$\begin{pmatrix} 1 & 1 & 2 & -3 \\ 3 & 5 & 10 & -4 \\ -2 & 2 & 1 & 18 \\ 1 & -1 & 4 & -12 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 2 & -3 \\ 0 & 2 & 4 & 5 \\ 0 & 4 & 5 & 12 \\ 0 & -2 & 2 & -9 \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} 1 & 1 & 2 & -3 \\ 0 & 2 & 4 & 5 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 6 & -4 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 2 & -3 \\ 0 & 2 & 4 & 5 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where the row operations are R_2-3R_1 R_3+2R_1, R_4-R_1, then R_3-2R_2, R_4+R_2, and finally R_4+2R_3. There is a row of zeros; this means that the vectors are linearly dependent. [5 marks]

(ii) The three nonzero rows of the reduced matrix in (i) have the same span as the four given vectors, and they are linearly independent because this process always results in linearly independent vectors. So $\mathbf{v}_1=(1,1,2,-3)$ and $\mathbf{w}_1=(0,2,4,5)$, $\mathbf{w}_2=(0,0,-3,2)$ are suitable vectors.

We can extend these to a basis for $\mathbf{R}^4$ by adding any row to the matrix with $\mathbf{v}_1$, $\mathbf{w}_1$, $\mathbf{w}_2$ as rows in such a way as to give four independent rows. The simplest thing is to add $(0,0,0,1)$, which gives a matrix in row-echelon form with no zero row. So we can extend to a basis of $\mathbf{R}^4$ by adding the vector $(0,0,0,1)$. [2 marks for the independent vectors spanning S, 3 for completing to a basis]

(iv) We need only check whether there are scalars a,b,c such that $a(1,1,2,-3)+b(0,2,4,5)+c(0,0,-3,2)=(-2,4,2,23)$. From the first components we have $a=-2$ and from the second components it follows that $-2+2b=4$ so that $b=3$. Then from the third components we get $-4+12-3c=2$, giving $c=2$ and using these values for a,b,c in the fourth components we get $6+15+4=23$ which is a contradiction. Hence the vector $(-2,4,2,23)$ does not lie in S. [5 marks]