SECTION A

1. Let $z = 5 - i$. Find the real and imaginary parts of $\overline{z} + \dfrac{2}{z}$. [4 marks]

2. Let $z = -3 + 3i$. Express z in the form $re^{i\theta}$. (As usual, $r > 0$ and θ is real.) Indicate the position of z on an Argand diagram. Use de Moivre's theorem to find the real and imaginary parts of z^4. [6 marks]

3. Verify that $(3-5i)^2 = -16-30i$. By means of the quadratic formula, or completing the square, solve the quadratic equation

$$z^2 + (1-3i)z + (2+6i) = 0.$$

[5 marks]

4. Let A, B, C be three points with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ respectively. Let P be the centroid of the triangle ABC, which has position vector $\mathbf{p} = \frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$. (You need not show this.)

Show that $\vec{PA} + \vec{PB} + \vec{PC}$ is the zero vector. [4 marks]

5. Let $A = (2, 0, 1)$, $B = (1, 2, -3)$ and $C = (-2, 3, -4)$.

(i) Find the vectors $\vec{AB}$, $\vec{AC}$ and $\vec{AB} \times \vec{AC}$. Verify that your vector $\vec{AB} \times \vec{AC}$ is perpendicular to $\vec{AB}$ and $\vec{AC}$. Write down the area of the triangle ABC. [5 marks]

(ii) Find the cosine of the angle BAC. (You need not evaluate any square roots which arise.) [2 marks]

(iii) Find an equation for the plane containing the triangle ABC. [3 marks]

6. Find the values of p, q, r such that the curve $y = p + qx + rx^2$ passes through the points $(1, 0), (-1, 6)$ and $(2, 12)$. [5 marks]

7. For each set of vectors (a) and (b) decide, giving reasons, whether the vectors are linearly independent and also whether they span $\mathbf{R}^3$.

$$\text{(a)}\quad \mathbf{u} = (1, -5, -8),\ \ \mathbf{v} = (-2, 10, -16),$$

$$\text{(b)}\quad \mathbf{u} = (1, -5, -8),\ \ \mathbf{v} = (-2, 10, -16),\ \ \mathbf{w} = (-1, 5, -56).$$

If the vectors in (a) or (b) are linearly *dependent*, find a non-trivial linear combination equalling the zero vector. [7 marks]

8. Find the determinants of the matrices A and B:

$$A = \begin{pmatrix} 1 & -2 & 4 \\ 0 & 2 & 5 \\ 1 & -2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -2 & 0 \\ 0 & 1 & -6 \\ 0 & 0 & 4 \end{pmatrix}.$$

Use the rules for determinants to *write down* the determinants of A^2B^{-1} and $B - I$, where I is the 3×3 identity matrix. [6 marks]

9. Find the eigenvalues of the matrix $A = \begin{pmatrix} 1 & -2 \\ -3 & 0 \end{pmatrix}$. [3 marks]

10. Let

$$B = \begin{pmatrix} 1 & 0 & 4 \\ 1 & 2 & -2 \\ 2 & 3 & -1 \end{pmatrix}.$$

Find a nonzero vector $\mathbf{v} = (x, y, z)^\top$ satisfying $(B - 3I)\mathbf{v} = \mathbf{0}$, where I is the 3×3 identity matrix. Which real number λ is therefore an eigenvalue of B? Write down a corresponding *unit length* eigenvector. [5 marks]

SECTION B

11. Express the complex number $a = -4 - 4\sqrt{3}\,i$ in the form $|a|e^{i\alpha}$. Find all the solutions of the equation $z^3 = a$ in the form $z = re^{i\theta}$ and indicate their positions clearly on an Argand diagram. For *one* of the solutions, express it in the cartesian form $z = x + iy$, correct to two decimal places.

Write down, with a brief explanation, *one* solution w to the equation $w^4 = \overline{a}$, in cartesian form correct to two decimal places. [15 marks]

12. Let

$$A = \begin{pmatrix} 1 & \alpha & 2 \\ -3 & 3 & \alpha - 1 \\ 3 & -3 & 5 \end{pmatrix}.$$

(i) Show that A is invertible if and only if $\alpha \neq -1$ and $\alpha \neq -4$. [5 marks]

(ii) Find the inverse of A when $\alpha = 0$. [5 marks]

(iii) Find a condition which a, b and c must satisfy for the system of equations

$$\begin{array}{rcrcrcl} x & - & y & + & 2z & = & a \\ -3x & + & 3y & - & 2z & = & b \\ 3x & - & 3y & + & 5z & = & c \end{array}$$

to be consistent. [5 marks]

13. Let L denote the line of intersection of the planes in $\mathbf{R}^3$ with equations

$$x + y + z = 5 \quad \text{and} \quad 2x - y + 4z = 8.$$

Let L' denote the line joining the points $A = (1, 2, 3)$ and $B = (-9, 6, 7)$.

(i) Find in parametric form an expression for the general point of L. [4 marks]

(ii) Write down the vector $\vec{AB}$ and an expression for the general point of L'. [3 marks]

(iii) Determine the point at which L' meets the plane

$$x + 2y + z = 10.$$

[3 marks]

(iv) Show that L meets L' and find the point of intersection. [5 marks]

14. Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ in $\mathbf{R}^4$ are defined by

$$\mathbf{v}_1 = (1, 2, -3, -2),\ \mathbf{v}_2 = (3, 1, 0, -1),\ \mathbf{v}_3 = (-3, 4, -9, -4),\ \mathbf{v}_4 = (-1, 3, -6, -3).$$

(i) Show that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly dependent. [5 marks]

(ii) Let S be the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$. Find linearly independent vectors with the same span S. Extend these linearly independent vectors to a basis of $\mathbf{R}^4$. [5 marks]

(iii) Decide whether the vector $(-1, -7, 12, 2)$ lies in S. [5 marks]