Solutions to MATH103 January 2004 Examination

Note: all questions are similar to homework or examples done in class.

Section A

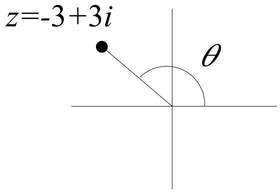
1. $z = 5 - i$ gives $\overline{z} = 5 + i$ so the required expressions is

$$5 + i + \frac{2}{5-i} \times \frac{5+i}{5+i} = 5 + i + \frac{10+2i}{5^2+1^2} = 5 + i + \frac{5+i}{13},$$

and the real part is $5 + \frac{5}{13} = \frac{70}{13}$ while the imaginary part is $1 + \frac{1}{13} = \frac{14}{13}$.

[1 mark for $\overline{z}$, 3 marks for calculation.]

2. $|z| = \sqrt{(-3)^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$ [1 mark]
Let $\theta = \arg(z)$. Then $\tan\theta = \frac{3}{-3} = -1$, so $\theta = -\frac{\pi}{4}$ or $\frac{3\pi}{4}$. Clearly from the diagram, $\frac{3\pi}{4}$ is correct. Thus $z = 3\sqrt{2}e^{3\pi i/4}$.



[1 mark for diagram, 2 marks for arg of z]

By de Moivre's theorem,

$$z^4 = (3\sqrt{2})^4 e^{4 \times \frac{3\pi i}{4}} = 324e^{3\pi i} = 324(-1) = -324,$$

since $e^{3\pi i} = e^{\pi i} = -1$. The real part of z^4 is –324 and the imaginary part is 0. [2 marks]

3. $(3 - 5i)^2 = 3^2 + 2 \times 3 \times (-5i) + (-5i)^2 = 9 - 30i - 25 = -16 - 30i$ [1 mark]
Thus the square roots of $-16 - 30i$ are $\pm(3 - 5i)$. Using the quadratic formula,

$$z = \frac{3i - 1 \pm \sqrt{(3i-1)^2 - 4(2+6i)}}{2} = \frac{3i - 1 \pm \sqrt{-9 - 6i + 1 - 8 - 24i}}{2}$$

$$= \frac{3i - 1 \pm \sqrt{-16 - 30i}}{2} = \frac{3i - 1 \pm (3-5i)2}{2} = 1 - i \text{ or } -2 + 4i.$$

[4 marks]

4. $\vec{PA} = \mathbf{p} - \mathbf{a} = \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c}) - \mathbf{a} = \frac{1}{3}(\mathbf{b} + \mathbf{c} - 2\mathbf{a})$. Similiarly $\vec{PB} = \frac{1}{3}(\mathbf{c} + \mathbf{a} - 2\mathbf{b})$, $\vec{PC} = \frac{1}{3}(\mathbf{a} + \mathbf{b} - 2\mathbf{c})$. Adding these three vectors all the terms cancel, giving the zero vector. [4 marks]

5. (i) $\vec{AB} = (1-2, 2-0, -3-1) = (-1, 2, -4)$, $\vec{AC} = (-4, 3, -5)$, [1 mark]
$\vec{AB} \times \vec{AC} = (2(-5) - (-4)(3), (-4)(-4) - (-1)(-5), (-1)(3) - 2(-4)) = (2, 11, 5)$.

To show this is perpendicular to $\vec{AB}$ we form the dot (scalar) product of these two vectors: $(2, 11, 5) \cdot (-1, 2, -4) = -2 + 22 - 20 = 0$.

Similarly to show $\vec{AB} \times \vec{AC}$ is perpendicular to $\vec{AC}$ we form the dot product of these two vectors: $(2, 11, 5) \cdot (-4, 3, -5) = -8 + 33 - 25 = 0$. [3 marks]

The area of the triangle is $\frac{1}{2}| \vec{AB} \times \vec{AC} | = \frac{1}{2}\sqrt{2^2 + 11^2 + 5^2} = \frac{1}{2}\sqrt{150}$. [1 mark]

(ii) The cosine of angle BAC is $\dfrac{\vec{AB} \cdot \vec{AC}}{| \vec{AB} | \, | \vec{AC} |} = \dfrac{4 + 6 + 20}{\sqrt{21}\sqrt{50}} = \dfrac{30}{\sqrt{1050}}$. [2 marks]

(iii) A normal to the plane ABC is given by $\mathbf{n} = \vec{AB} \times \vec{AC} = (2, 11, 5)$ as above. An equation is then

$$(\mathbf{x} - \mathbf{a}) \cdot \mathbf{n} = 0, \text{ that is } (x - 2, y, z - 1) \cdot (2, 11, 5) = 0, \text{ giving } 2x + 11y + 5z = 9.$$

(Alternatively, the plane will be $2x + 11y + 5z = k$ for some number k, since its normal is $(2, 11, 5)$ and substituting the coordinates of A (or B or C) in the equation gives $k = 9$.)
[3 marks]

6. We have to solve the equations

$$\begin{aligned} p + q + r &= 0 \qquad (1)\\ p - q + r &= 6 \qquad (2)\\ p + 2q + 4r &= 12 \qquad (3) \end{aligned}$$

(1)–(2) gives $2q = -6$ so $q = -3$. Then (3)–(1) gives $q + 3r = 12$ so $3r = 12 - q = 15$ giving $r = 5$. Finally (1) gives $p = -2$.
[2 marks for setting up the equations and 3 for solving them]

7. (a) $\mathbf{v} = (1, -5, -8)$, $\mathbf{w} = (-2, 10, -16)$ are *linearly independent* since the second is not a scalar multiple of the first (note that the first two components of $\mathbf{w}$ are –2 times the first two components of $\mathbf{v}$, but the third component of $\mathbf{w}$ is +2 times the third component of $\mathbf{v}$). There are only two vectors so they *cannot span* $\mathbf{R}^3$. [2 marks]

(b) Putting the vectors, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ say, as the rows of a matrix and using row reduction gives

$$\begin{pmatrix} 1 & -5 & -8 \\ -2 & 10 & -16 \\ -1 & 5 & -56 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -5 & -8 \\ 0 & 0 & -32 \\ 0 & 0 & -64 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -5 & -8 \\ 0 & 0 & -32 \\ 0 & 0 & 0 \end{pmatrix},$$

using $R_2 + 2R_1$, $R_3 + R_1$, and then $R_3 - 2R_2$. The row of zeros shows that the three vectors are *linearly dependent*. Since they are linearly dependent they *do not span* $\mathbf{R}^3$.
[3 marks]

The rows of the second matrix are $\mathbf{u}$, $\mathbf{v} + 2\mathbf{u}$, $\mathbf{w} + \mathbf{u}$ respectively, and we therefore have $\mathbf{w} + \mathbf{u} = 2(\mathbf{v} + 2\mathbf{u})$, which gives $3\mathbf{u} + 2\mathbf{v} - \mathbf{w} = 0$. [2 marks]

8. Evaluating $\det(A)$ by the first column gives
$\det(A) = 1(2(-1) - 5(-2)) + 1((-2)(5) - 4(2)) = 8 - 18 = -10$. [3 marks]

For B we need only multiply the diagonal entries, since B is upper triangular. So $\det(B) = 8$. [1 mark]

Using rules for determinants we now get $\det(A^2B^{-1}) = (\det(A))^2/\det(B) = \frac{100}{8} = \frac{25}{2}$.
The matrix $B - I$ is also upper triangular, and has diagonal entries 1, 0, 3. The determinant is the product of these, hence is 0. [2 marks]

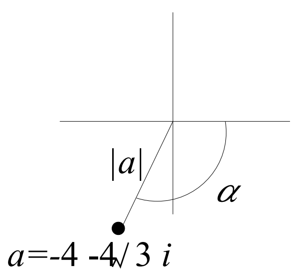
9. The eigenvalues of A are the solutions of the equation $\det(A - \lambda I) = \begin{pmatrix} 1-\lambda & -2 \\ -3 & -\lambda \end{pmatrix} = 0$.
This gives $(1-\lambda)(-\lambda) - (-2)(-3) = 0$, that is $\lambda^2 - \lambda - 6 = 0$, which factorises as $(\lambda - 3)(\lambda + 2) = 0$ (or solve by the quadratic formula), so that the eigenvalues are 3 and –2. [3 marks]

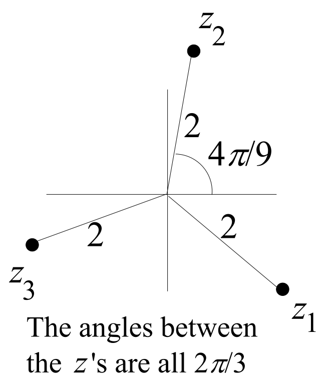
10. $B - 3I = \begin{pmatrix} -2 & 0 & 4 \\ 1 & -1 & -2 \\ 2 & 3 & -4 \end{pmatrix}$ and we want to solve the equations in x, y, z with this coefficient matrix. It is clear by adding rows 1 and 3 that $y = 0$. Then using row 1 we get $x = 2z$, giving a solution $(2, 0, 1)$ which also satisfies equation obtained from the second row. Having found a nontrivial solution it follows that 3 is an eigenvalue of B. (More generally, if there is a nonzero solution of $\det(B - \lambda I) = 0$, then λ is an eigenvalue.) [3 marks]

A unit eigenvector is $\dfrac{(2,0,1)}{\sqrt{2^2 + 0^2 + 1^2}} = \dfrac{(2,0,1)}{\sqrt{5}}$. [2 marks]

Section B

11. $|a| = \sqrt{(-4)^2 + (-4\sqrt{3})^2} = \sqrt{16 + 48} = 8$. Next, $\tan\alpha = \frac{-4\sqrt{3}}{-4} = \sqrt{3}$ and one solution for this is $\alpha = \frac{\pi}{3}$, the other being $\frac{\pi}{3} - \pi = -\frac{2\pi}{3}$, From the left-hand diagram, the latter is clearly the correct value. [4 marks]





Now write $z = re^{i\theta}$, giving $z^3 = r^3e^{3i\theta}$ and equating this to $a = 8e^{-2\pi i/3}$ gives
$r^3 = 8$, so that, r being real and > 0, we have $r = 2$,
$3\theta = -\frac{2\pi}{3} + 2k\pi$, where we take $k = 0, 1, 2$ for the distinct solutions (for $z^n = a$ we take $k = 0, 1, \ldots, n-1$).
Hence, $\theta = -\frac{2\pi}{9} + \frac{2k\pi}{3}$, that is, $\theta = -\frac{2\pi}{9}, \frac{4\pi}{9}, \frac{10\pi}{9}$ (the last can be replaced by $\frac{10\pi}{9} - 2\pi = -\frac{8\pi}{9}$ if the principal value, between $-\pi$ and π, is wanted). The solutions
$z_1 = 2e^{-2\pi i/9}$, $z_2 = 2e^{4\pi i/9}$, $z_3 = 2e^{10\pi i/9} = 2e^{-8\pi i/9}$ are indicated approximately on the right-hand diagram above. [4 marks for the solution, 3 for the diagram]

Using a calculator, the solution $z_1 = 2(\cos(-\frac{2\pi}{9}) + i\sin(-\frac{2\pi}{9})) = 1.53 - 1.29i$ to two decimal places. [2 marks]

The solutions of $w^3 = \overline{a}$ are the conjugates of the solutions of $z^3 = a$, since $z^3 = a$ gives $\overline{a} = \overline{z^3} = \overline{z}^3$. Therefore one solution is $w = 1.53 + 1.29i$. [2 marks]

12. (i) We have: A is invertible if and only if $\det(A) \neq 0$. Calculating $\det(A)$, it is worth adding rows 2 and 3 first (this row operation does not alter the determinant), giving $\begin{pmatrix} 1 & \alpha & 2 \\ -3 & 3 & \alpha - 1 \\ 0 & 0 & \alpha + 4 \end{pmatrix}$. Evaluating by the third row now gives $\det(A) = (\alpha + 4)(3 + 3\alpha) = 3(\alpha + 4)(\alpha + 1)$. Hence A is invertible if and only if $\alpha \neq -4, -1$ as required.

[4 marks for evaluating the determinant, 1 for deducing when A is invertible]

(ii) We need the inverse of $A_0 = \begin{pmatrix} 1 & 0 & 2 \\ -3 & 3 & -1 \\ 3 & -3 & 5 \end{pmatrix}$. The matrix of cofactors of A_0 is $\begin{pmatrix} 12 & 12 & 0 \\ -6 & -1 & 3 \\ -6 & -5 & 3 \end{pmatrix}$ and the determinant of A_0 is 12, using the formula found in (i). Transposing the matrix of cofactors and dividing by the determinant we get

$A_0^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & -\frac{1}{12} & -\frac{5}{12} \\ 0 & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$ [5 marks]

(iii) Using row operations on the augmented matrix of the equations we get

$$\begin{pmatrix} 1 & -1 & 2 & a \\ -3 & 3 & -2 & b \\ 3 & -3 & 5 & c \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 2 & a \\ 0 & 0 & 4 & b + 3a \\ 0 & 0 & -1 & c - 3a \end{pmatrix}$$

using $R_2 + 3R_1,\ R_3 - 3R_1$. Now a further operation $4R_3 + R_2$ gives a third row $(0, 0, 0, -9a + b + 4c)$, so that the required condition is $-9a + b + 4c = 0$. [5 marks]

13. (i)

$$\begin{aligned} x + y + z &= 5 \qquad (4) \\ 2x - y + 4z &= 8 \qquad (5) \end{aligned}$$

Taking $(5) - 2 \times (4)$ gives $-3y + 2z = -2$, that is $z = \frac{1}{2}(-2 + 3y) = -1 + \frac{3}{2}y$. Substituting in (4) gives $x = 6 - \frac{5}{2}y$, so a parametric form of L is $(6 - \frac{5}{2}y, y, -1 + \frac{3}{2}y)$. (It is equally valid to parametrize by z; this gives $(\frac{13}{3} - \frac{5}{3}z, \frac{2}{3} + \frac{2}{3}z, z)$. It is also valid to parametrize by x.) [4 marks]

(ii) $\overrightarrow{AB} = (-10, 4, 4)$ so the general point of L' is
$(1, 2, 3) + \lambda(-10, 4, 4) = (1 - 10\lambda, 2 + 4\lambda, 3 + 4\lambda)$. [3 marks]

(iii) L' meets the plane $x + 2y + z = 10$ in the point whose parameter λ is obtained by substituting the general point as in (ii) into the equation of the plane. This gives $1 - 10\lambda + 2(2 + 4\lambda) + 3 + 4\lambda = 10$, that is $2\lambda + 8 = 10$, giving $\lambda = 1$. Thus the point of intersection is $(-9, 6, 7)$. [3 marks]

(iv) In order for L to meet L' we need there to exist values of y and λ which make the general points in (i) and (ii) equal. Thus we require that the three equations

$1-10\lambda = 6-\frac{5}{2}y,\ \ 2+4\lambda = y,\ \ 3+4\lambda = -1+\frac{3}{2}y$ should have a common solution for y and λ. Rearranging the equations gives, respectively, $y-4\lambda = 2,\ \ y-4\lambda = 2,\ \ 3y-8\lambda = 8$. Two of these equations are the same, and solving gives $y = 4,\ \lambda = \frac{1}{2}$ which satisfies all three. Hence the two lines do meet, in the point obtained by substituting $y = 4$ into the general point in (i) or $\lambda = \frac{1}{2}$ into the general point in (ii). These give $(-4, 4, 5)$ as the point of intersection of L and L'. [5 marks]

14. (i) Writing the vectors as the rows of a matrix and using row reduction gives

$$\begin{pmatrix} 1 & 2 & -3 & -2 \\ 3 & 1 & 0 & -1 \\ -3 & 4 & -9 & -4 \\ -1 & 3 & -6 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & -3 & -2 \\ 0 & -5 & 9 & 5 \\ 0 & 10 & -18 & -10 \\ 0 & 5 & -9 & -5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & -3 & -2 \\ 0 & -5 & 9 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where the row operations are $R_2 - 3R_1\ R_3 + 3R_1,\ R_4 + R_1$ and then $R_3 + 2R_2,\ R_4 + R_2$. There is at least one row of zeros; this means that the vectors are linearly dependent. [5 marks]

(ii) The two nonzero rows of the reduced matrix in (i) have the same span as the given vectors, and they are linearly independent because this process always results in linearly independent vectors. So $\mathbf{v}_1 = (1, 2, -3, -2)$ and $\mathbf{w} = (0, -5, 9, 5)$ are suitable vectors. We can extend these to a basis for $\mathbf{R}^4$ by adding any two rows to the matrix with $\mathbf{v}_1,\ \mathbf{w}$ as rows in such a way as to give four independent rows. The simplest thing is to add $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$ which give an upper triangular matrix whose determinant is nonzero since the four diagonal entries are nonzero. [2 marks for the independent vectors spanning S, 3 for completing to a basis]

(iv) We need only check whether there are scalars λ, μ such that $\lambda(1, 2, -3, -2) + \mu(0, -5, 9, -5) = (-1, -7, 12, 2)$. From the first components we have $\lambda = -1$ and from the second components it follows that $2\lambda - 5\mu = -7$ so $\mu = 1$. The third components give $-3\lambda + 9\mu = 12$ which is true for these λ and μ but the fourth components give $-2\lambda - 5\mu = 2$ which is incorrect. So $(-1, -7, 12, 2)$ does not belong to S. [5 marks]