SECTION A

1. Let $z = 3-2i$. Find the real and imaginary parts of $1 + \dfrac{1}{z^2}$. [4 marks]

2. Let $z = -\sqrt{3} - i$. Express z in the form $re^{i\theta}$. (As usual, $r > 0$ and θ is real.) Indicate the position of z on an Argand diagram. Use de Moivre's theorem to find the real and imaginary parts of z^6. [6 marks]

3. Verify that $(7+4i)^2 = 33 + 56i$. By means of the quadratic formula, or completing the square, solve the quadratic equation

$$z^2 + (-3+2i)z - (7+17i) = 0.$$ [5 marks]

4. Let A, B, C be three points with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ respectively. Write down the following position vectors:
(i) $\mathbf{p}$, for P which is on BA, two-thirds of the distance from B to A,
(ii) $\mathbf{q}$, for Q which is on CA, two-thirds of the distance from C to A,
(iii) $\mathbf{r}$, for R which is the midpoint of BC, and
(iv) $\mathbf{s}$, for S which is the midpoint of PQ.

Find a scalar λ such that $\mathbf{s} = \lambda\mathbf{a} + (1-\lambda)\mathbf{r}$. What can you deduce about the three points A, S and R? [7 marks]

5. Let $A = (1,-1,1),\ B = (2,2,2)$ and $C = (4,3,-1)$.

(i) Find the vectors $\vec{AB}$, $\vec{AC}$ and $\vec{AB} \times \vec{AC}$. Write down the area of the triangle ABC. [4 marks]

(ii) Find the length of the perpendicular from B to the side AC. [2 marks]

(iii) Find an equation for the plane containing the triangle ABC. [3 marks]

6. Find the values of p, q, r such that the curve $y = p + qx + rx^2$ passes through the points $(1,5),\ (2,4)$ and $(-1,-5)$. [5 marks]

7. For each set of vectors (a) and (b) decide, giving reasons, whether the vectors are linearly independent and also whether they span $\mathbf{R}^3$.

(a) $(4,-1,-4),\ (-8,2,8),$ (b) $(4,-1,-4),\ (8,1,1),\ (-4,0,2).$

[5 marks]

8. Find the determinants of the matrices A and B:

$$A = \begin{pmatrix} 1 & 1 & -3 \\ 2 & 4 & 0 \\ 1 & 6 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 0 & 0 \\ 1 & -1 & 0 \\ 6 & 2 & 4 \end{pmatrix}.$$

Write down the determinants of BA^{-1} and $B + I$, where I is the 3×3 identity matrix. [6 marks]

9. Find the eigenvalues of the matrix $A = \begin{pmatrix} 3 & 5 \\ 1 & -1 \end{pmatrix}$. [3 marks]

10. Let

$$B = \begin{pmatrix} 2 & -3 & 5 \\ 1 & -2 & 4 \\ 1 & -3 & 2 \end{pmatrix}.$$

Find a nonzero vector $\mathbf{v} = (x, y, z)^\top$ satisfying $(B - I)\mathbf{v} = \mathbf{0}$. Which real number λ is therefore an eigenvalue of B? Write down a corresponding *unit length* eigenvector. [5 marks]

SECTION B

11. Express the complex number $a = -8\sqrt{2} - 8\sqrt{2}\,i$ in the form $|a|e^{i\alpha}$. Find all the solutions of the equation $z^4 = a$ in the form $z = re^{i\theta}$ and indicate their positions clearly on an Argand diagram. For *one* of the solutions, express it in the cartesian form $z = x + iy$, correct to two decimal places. Explain briefly how to use this solution to obtain the other three solutions in cartesian form.

Write down, with a brief explanation, *one* solution w to the equation $w^4 = \overline{a}$. [15 marks]

12. Let

$$A = \begin{pmatrix} 3 & -3 & \alpha \\ -1 & 3-\alpha & 3 \\ 2 & -1 & 4 \end{pmatrix}.$$

(i) Show that A is invertible if and only if $\alpha \neq \frac{15}{2}$ and $\alpha \neq 1$. [5 marks]

(ii) Find the inverse of A when $\alpha = 3$. [5 marks]

(iii) Find a condition which a, b and c must satisfy for the system of equations

$$\begin{array}{rcrcrcl} 3x & - & 3y & + & z & = & a \\ -x & + & 2y & + & 3z & = & b \\ 2x & - & y & + & 4z & = & c \end{array}$$

to be consistent. [5 marks]

13. Let L denote the line of intersection of the planes in $\mathbf{R}^3$ with equations

$$2x - y + 3z = 5 \quad \text{and} \quad x + 4y + z = 6.$$

Let L' denote the line joining the points $A = (2, 3, -4)$ and $B = (1, 0, 5)$.

(i) Find in parametric form an expression for the general point of L. [4 marks]

(ii) Write down the vector $\vec{AB}$ and an expression for the general point of L'. [3 marks]

(iii) Determine the point at which L' meets the plane

$$x + y + z = 11.$$

[4 marks]

(iv) Decide whether or not L meets L'. [4 marks]

14. Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ in $\mathbf{R}^4$ are defined by

$$\mathbf{v}_1 = (1, 3, 5, -2), \; \mathbf{v}_2 = (2, 2, 1, 0), \; \mathbf{v}_3 = (1, 1, 2, 0), \; \mathbf{v}_4 = (2, 4, 4, -2).$$

(i) Show that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly dependent. [5 marks]

(ii) Let S be the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$. Find linearly independent vectors with the same span S. Extend these linearly independent vectors to a basis of $\mathbf{R}^4$. [5 marks]

(iii) Show that the vector $(2, 8, 1, -6)$ lies in S. [5 marks]