SECTION A

1. Let $z = 1 + 4i$. Find the real and imaginary parts of $\overline{z} + \frac{2}{z}$. [4 marks]

2. Let $z = 2\sqrt{3} - 2i$. Express z in the form $re^{i\theta}$. (As usual, $r > 0$ and θ is real.) Indicate the position of z on an Argand diagram. Use de Moivre's theorem to find the real and imaginary parts of z^3. [7 marks]

3. Verify that $(5+i)^2 = 24+10i$. By means of the quadratic formula, or completing the square, solve the quadratic equation $z^2+(1-3i)z-8-4i = 0$. [5 marks]

4. Let A, B, C, D be four points with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ respectively. Write down the position vectors of the mid-point P of AB; the mid-point Q of BC; the mid-point R of CD and the mid-point S of DA. Show that the vector $\overrightarrow{PQ}$ equals the vector $\overrightarrow{SR}$. [5 marks]

5. Let $A = (-1, 1, 2)$, $B = (0, 2, 4)$ and $C = (-3, 1, 3)$.

(i) Find the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AB} \times \overrightarrow{AC}$. Write down the area of the triangle ABC. [3 marks]

(ii) Find the angle BAC. [3 marks]

(iii) Find an equation for the plane containing the triangle ABC. [3 marks]

6. Find the values of p, q, r such that the curve $y = p + qx + rx^2$ passes through the points $(0, 2)$, $(1, 0)$ and $(3, 2)$. [5 marks]

7. For each set of vectors (a) and (b) decide, giving reasons, whether the vectors are linearly independent and also whether they span $\mathbf{R}^3$.

(a) $(2, -1, 3)$, $(4, -2, 5)$, (b) $(2, -1, 3)$, $(4, -2, 5)$, $(-2, -1, 4)$.

[6 marks]

8. Find the determinants of the matrices A and B:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -3 & -4 \\ 2 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 10 & -12 \\ 0 & 4 & -6 \\ 0 & 0 & -2 \end{pmatrix}.$$

Write down the determinants of BA^{-1} and $B + I$, where I is the 3×3 identity matrix. [6 marks]

9. Find the eigenvalues of the matrix $A = \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix}$. [3 marks]

10. Let

$$B = \begin{pmatrix} 1 & -2 & 1 \\ -1 & 2 & -3 \\ 2 & 2 & 0 \end{pmatrix}.$$

Find a nonzero vector $\mathbf{v} = (x, y, z)^\top$ satisfying $(B - 3I)\mathbf{v} = \mathbf{0}$. Deduce that $\lambda = 3$ is an eigenvalue of B, and write down a corresponding *unit length* eigenvector. [5 marks]

SECTION B

11. Express the complex number $a = -4\sqrt{2} + 4i\sqrt{2}$ in the form $|a|e^{i\alpha}$. Find all the solutions of the equation $z^3 = a$ in the form $z = re^{i\theta}$ and indicate their positions clearly on an Argand diagram. For *one* of the solutions, express it in the cartesian form $z = x + iy$.

Without further calculation indicate clearly on a separate diagram the solutions of the equation $z^3 = \overline{a} = -4\sqrt{2} - 4i\sqrt{2}$. [15 marks]

12. Let

$$A = \begin{pmatrix} 2 & 1 & \alpha+1 \\ -1 & \alpha & 1 \\ 1 & -1 & 3 \end{pmatrix}.$$

(i) Show that A is invertible if and only if $\alpha \neq -1$ and $\alpha \neq 7$. [5 marks]

(ii) Find the inverse of A when $\alpha = 0$. [5 marks]

(iii) Find a condition which a, b and c must satisfy for the system of equations

$$\begin{array}{rcrcrcl} 2x & + & y & + & 8z & = & a \\ -x & + & 7y & + & z & = & b \\ x & - & y & + & 3z & = & c \end{array}$$

to be consistent. [5 marks]

13. Let L denote the line of intersection of the planes in $\mathbf{R}^3$ with equations

$$x + y - z = 5 \quad \text{and} \quad 2x + 3y + 4z = 7.$$

Let L' denote the line joining the points $A = (1, 2, 3)$ and $B = (0, -2, 4)$.

(i) Find in parametric form an expression for the general point of L. [4 marks]

(ii) Write down the vector $\overrightarrow{AB}$ and an expression for the general point of L'. [3 marks]

(iii) Determine the point at which L' meets the plane

$$2x - y + z = 0.$$

[4 marks]

(iv) Decide whether or not L meets L'. [4 marks]

14. Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ in $\mathbf{R}^4$ are defined by

$$\mathbf{v}_1 = (1, 0, -2, 3),\ \mathbf{v}_2 = (3, 1, 0, -2),\ \mathbf{v}_3 = (2, 0, 1, 1),\ \mathbf{v}_4 = (2, 1, -3, 0).$$

(i) Show that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly dependent. [5 marks]

(ii) Let S be the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$. Find linearly independent vectors with the same span S. Extend these linearly independent vectors to a basis of $\mathbf{R}^4$. [5 marks]

(iii) Show that the vector $(6, 1, -1, 2)$ lies in S. [5 marks]