MATH103 January 2001 Examination

All answers from Section A and the best three answers from Section B will be taken into account.

SECTION A

1. Let $z = 1 - 3i$. Find the real and imaginary parts of $z + \frac{1}{z^2}$. [4 marks]

2. Let $z = -3 + i\sqrt{3}$. Express z in the form $re^{i\theta}$. (As usual, $r > 0$ and θ is real.) Indicate the position of z on an Argand diagram. Use de Moivre's theorem to find the real and imaginary parts of z^6. [6 marks]

3. Verify that $(-4+5i)^2 = -9 - 40i$. By means of the quadratic formula, or completing the square, solve the quadratic equation $z^2 - iz + 2 + 10i = 0$.
[5 marks]

4. Let A, B, C be three points with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ respectively. Write down the position vector of the point P which is on the segment AB, three-quarters of the distance from A to B. Write down the position vector of the point Q which is on the segment CB, three-quarters of the distance from C to B. Find the vector $\vec{PQ}$ and verify that it is $\frac{1}{4}\,\vec{AC}$. [4 marks]

5. Let $A = (1, 2, 0)$, $B = (-1, 3, 5)$ and $C = (2, -1, 3)$.

(i) Find the vectors $\vec{AB}$, $\vec{AC}$ and $\vec{AB} \times \vec{AC}$. Find the area of the triangle ABC. [3 marks]

(ii) Find the length of the altitude of the triangle ABC drawn through the vertex A. [2 marks]

(iii) Find an equation for the plane containing the triangle ABC.
[2 marks]

6. Find an equation for the plane which has normal vector $\mathbf{n} = (2, -1, 3)$ and which passes through the point $(2, 0, 1)$. Find the distance of this plane from the point $(3, 1, -6)$. [5 marks]

7. Find the values of p, q, r such that the curve $y = p + qx + rx^2$ passes through the points $(1, 6)$, $(-1, 8)$ and $(2, 17)$. [4 marks]

8. For each set of vectors (a) and (b) decide, giving reasons, whether the vectors are linearly independent and whether they span $\mathbf{R}^3$.

(a) $(1, 0, 1)$, $(3, 2, -1)$; (b) $(1, 0, 1)$, $(3, 2, -1)$, $(1, -2, 3)$.

[5 marks]

9. Find the determinants of the matrices A and B:

$$A = \begin{pmatrix} 2 & 1 & 5 \\ 4 & 3 & -6 \\ 1 & 2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 3 & -7 \\ 0 & -3 & 5 \\ 0 & 0 & 7 \end{pmatrix}.$$

Write down the determinants of $A^{-1}B$ and $B + 3I$, where I is the 3×3 identity matrix. [6 marks]

10. Let

$$A = \begin{pmatrix} -1 & 10 & 6 \\ -2 & 7 & 4 \\ 1 & -1 & 0 \end{pmatrix}.$$

For each of the values $\lambda = 1, 2$ and 3, find a nonzero vector $\mathbf{v} = (x, y, z)^\top$ satisfying $(A - \lambda I)\mathbf{v} = \mathbf{0}$. Deduce that $\lambda = 1, 2, 3$ are the eigenvalues of A, and write down corresponding *unit length* eigenvectors. [9 marks]

SECTION B

11. Express the complex number $a = -8 - 8i\sqrt{3}$ in the form $|a|e^{i\alpha}$. Find all the solutions of the equation $z^4 = a$ in the form $z = re^{i\theta}$ and indicate their positions clearly on an Argand diagram. For *one* of the solutions, express it in the cartesian form $z = x + iy$.

Without further calculation indicate clearly on a separate diagram the solutions of the equation $z^4 = \overline{a} = -8 + 8i\sqrt{3}$, giving a brief reason for your answer. [15 marks]

12. Let

$$A = \begin{pmatrix} 2 & 7 & \alpha - 1 \\ 1 & 3 & -1 \\ -1 & \alpha & 0 \end{pmatrix}.$$

(i) Show that A is invertible if and only if $\alpha \neq -2$. [5 marks]

(ii) Find the inverse of A when $\alpha = 0$. [5 marks]

(iii) Find a condition which a, b and c must satisfy for the system of equations

$$\begin{array}{rcrcrcl} 2x & + & 7y & - & 3z & = & a \\ x & + & 3y & - & z & = & b \\ -x & - & 2y & & & = & c \end{array}$$

to be consistent. [5 marks]

13. Let L denote the line of intersection of the planes in $\mathbf{R}^3$ with equations

$$x - y + 2z = 3 \quad \text{and} \quad 2x + y + z = 0.$$

Let L' denote the line joining the points $A = (-1, 1, 1)$ and $B = (-2, 3, 1)$.

(i) Find in parametric form an expression for the general point of L. [4 marks]

(ii) Write down the vector $\vec{AB}$ and an expression for the general point of L'. [3 marks]

(iii) Determine the point at which L' meets the plane

$$x - y + 2z = 3.$$

[4 marks]

(iv) Decide whether or not L meets L'. [4 marks]

14. Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ in $\mathbf{R}^4$ are defined by

$$\mathbf{v}_1 = (1, -1, 3, 1),\ \mathbf{v}_2 = (2, 1, -1, 2),\ \mathbf{v}_3 = (3, 0, 2, 4),\ \mathbf{v}_4 = (2, -2, 6, 4).$$

(i) Show that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly dependent. [5 marks]

(ii) Let S be the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$. Find linearly independent vectors with the same span S. Extend these linearly independent vectors to a basis of $\mathbf{R}^4$. [5 marks]

(iii) Show that the vector $(2, 4, -8, 5)$ lies in S. [5 marks]