SECTION A

1. Find all the solutions, if there are any, to the equations

$$\begin{aligned} x - 2y - 3z \phantom{{}-2t} &= 1 \\ 2x - y + z - 2t &= 0 \\ -x + 2y + 2z - t &= 1 \end{aligned}$$

[5 marks]

2. Find the values of p, q and r for which the curve $y = p + qx + rx^2$ passes through the points $(-2, 9), (1, -3)$ and $(2, 1)$. [5 marks]

3. Find the determinants of the matrices

$$A = \begin{pmatrix} 3 & 2 & 1 & 1 \\ 0 & -1 & 2 & 2 \\ 1 & 2 & -2 & 1 \\ 1 & 1 & -2 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 4 & 3 & 6 & 9 \\ 0 & -2 & 2 & 4 \\ 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & 6 \end{pmatrix}.$$

Write down the determinants of $B + 2I$ and AB^{-1}. [8 marks]

4. Simplify as far as you can the following matrix products, where A and B are square matrices of the same size and A is invertible.

$$(A^{-1}BA)^3, \quad A(A + 2I)A^{-1}, \quad (A - 2I)(A + 2I).$$

[4 marks]

5. Find the equation of the plane which is perpendicular to the vector $(1, 2, 3)$ and passes through the point $(-4, -5, 6)$. Find the distance of this plane from the point $(-1, -1, -1)$. [5 marks]

6. Let $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\mathbf{v} = \mathbf{i} + 3\mathbf{k}$, $\mathbf{w} = 3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$. Find $\mathbf{v} \times \mathbf{w}$ and $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$. Hence or otherwise find the volume of the parallelepiped which has one vertex at the point $A = (1, 2, 3)$ and three adjacent vertices at the points

$$B = (3, 3, 5),\ C = (2, 2, 6),\ D = (4, 1, 7).$$

[3 marks]

7. Find an orthonormal basis of $\mathbf{R}^3$ two vectors of which lie in the plane $2x - y + 3z = 0$. [5 marks]

8. Find the rank and nullity of the matrix

$$\begin{pmatrix} 1 & 2 & -1 & 3 & 2 & 0 \\ 1 & 3 & 2 & 2 & 0 & 1 \\ 0 & 2 & -1 & -2 & 3 & 4 \\ 2 & 3 & 2 & 7 & -1 & -3 \end{pmatrix}.$$

[5 marks]

9. Find the eigenvalues of the matrix

$$\begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}.$$

[3 marks]

10. Show that the matrix

$$\begin{pmatrix} 1 & -2 & 3 \\ -2 & 1 & 0 \\ -1 & -1 & 3 \end{pmatrix}$$

has eigenvalues 0, 2 and 3. For each eigenvalue, find a corresponding eigenvector. [9 marks]

SECTION B

11. Let

$$A = \begin{pmatrix} 1 & 3 & \alpha+1 \\ 2 & \alpha & 4 \\ 3 & 1 & 5 \end{pmatrix}.$$

(i) Show that A is invertible if and only if $\alpha \neq 2$ and $\alpha \neq -\frac{2}{3}$.

[5 marks]

(ii) Find the inverse of A when $\alpha = 1$. [5 marks]

(iii) Find a condition on a, b, c such that the system of equations

$$\begin{aligned} x + 3y + 3z &= a \\ 2x + 2y + 4z &= b \\ 3x + y + 5z &= c \end{aligned}$$

is consistent. [5 marks]

12. Let

$$A = (1,0,1),\ B = (2,-1,3),\ C = (1,4,0),\ D = (1,1,1),\ E = (2,3,2).$$

(i) Determine the equation of the plane passing through A, B and C. [4 marks]

(ii) Write down the vector $\overrightarrow{DE}$ and hence write down a parametrization of the line passing through D and E. [3 marks]

(iii) Find the point where the line DE meets the plane through A, B and C. [4 marks]

(iv) Determine whether the line DE meets the line AB. [4 marks]

13. Let S be the subspace of $\mathbf{R}^4$ spanned by the vectors

$$\mathbf{v} = (1, 2, -1, 1) \text{ and } \mathbf{w} = (2, 2, 1, -3).$$

Let T be the subspace of $\mathbf{R}^4$ defined by the two equations

$$x - y + z + t = 0 \text{ and } 2x - 5y + 2t = 0.$$

What are the dimensions of S and T?
Give brief reasons for your answers. [4 marks]

Show that $\lambda\mathbf{v} + \mu\mathbf{w}$ belongs to T if and only if $\lambda + 2\mu = 0$. Hence find a basis for $S \cap T$. [6 marks]

Extend this

(i) to a basis for S;

(ii) to a basis for T.

Extend your basis for T to a basis for $\mathbf{R}^4$. [5 marks]

14. The eigenvalues of the matrix

$$A = \begin{pmatrix} 1 & -2 & 2 \\ -2 & -4 & 1 \\ 2 & 1 & 4 \end{pmatrix}.$$

are $1, 5$ and -5.

(i) Find unit length eigenvectors corresponding to each of these eigenvalues. [8 marks]

(ii) Write down an orthogonal matrix P and a diagonal matrix D such that $P^{-1}AP = D$. [3 marks]

(iii) Express the quadric surface

$$x^2 - 4y^2 + 4z^2 - 4xy + 4xz + 2yz = -1$$

in standard form, and state the type of quadric. [4 marks]