JAN 2006

Answer all of section A and THREE questions from section B. The marks shown against questions, or parts of questions, indicate their relative weights. Section A carries 55% of the total marks.

SECTION A

1. Write down the natural domain and the range of each of the following functions and sketch their graphs:

(i) $y = |2x - 1|$ (ii) $y = 1/(3x+1)^2$. (4 marks)

2. If

$$f(x) = \frac{1-5x}{1+3x},$$

find the corresponding inverse function $f^{-1}(x)$. Verify that

$$f(f^{-1}(x)) = x.$$ (5 marks)

3. (i) State, giving reasons, whether or not the following function is odd, even or neither:

$$f(x) = \frac{x^5 - x}{x^7 - x^3 + 2x}.$$

(ii) Find the period of the function

$$g(x) = 3\sin(5x).$$ (4 marks)

4. Find the general solution of

$$\sqrt{3}\cos x - \sin x = \sqrt{2}$$ (6 marks)

5. (1) Given a function $f(x)$ defined as follows

$$f(x) = \begin{cases} x^2 + 2x + 3 & \text{for } x < 1 \\ mx + c & \text{for } x \geq 1 \end{cases}$$

find values of the constants m and c such that f is differentiable at $x = 1$. (4 marks)

6. Differentiate the following identifying any rules of differentiation that you use

(i) $y = t^5 \sec(3t)$ (ii) $y = \dfrac{1+3x}{1+x^2}$ (iii) $y = \cos(\exp(-x^2))$. (6 marks)

7. Use implicit differentiation to find expressions for dy/dx and d^2y/dx^2 in terms of x and y given that

$$xy + y^2 = -5x\,.$$ (6 marks)

8. Write down the definition of $\cosh x$ in terms of e^x and e^{-x}. Hence show that

$$\cosh^{-1}(x) = \ln\left(x + \sqrt{x^2 - 1}\right), \qquad x \geq 1\,.$$ (5 marks)

9. Find the following integrals

(i) $\displaystyle\int_0^1 \frac{e^t}{e^t + 5}\,dt$ (ii) $\displaystyle\int t^2 e^t\, dt\,.$ (6 marks)

10. The curve *C* has equation $y = 11x^5$, $0 \leq x \leq 1$. Find the volume of the solid of revolution generated when the finite region enclosed by *C*, the line $x = 1$ and the *x*-axis is rotated through 2π about the *x*-axis. (5 marks)

11. Use the integral test to determine whether or not the following infinite series converges

$$\sum_1^{\infty} \frac{1}{n^{5/2}}\,.$$ (4 marks)

SECTION B

12. (i) Given that $x = a$ satisfies the equation $f(x) = x$, verify that $x = a$ also satisfies the equation $f(f(x)) = x$.

(ii) The function *f* is defined by

$$f(x) = x(x-7).$$

Find solutions of the equation

$$f(x) = x.$$

Verify that $f(f(x)) = x^4 - 14x^3 + 42x^2 + 49x$ and use the result of (i) to find <u>all</u> the solutions of the equation $f(f(x)) = x$.

(iii) Apply the Newton-Raphson method with initial value $x_0 = -1.0$ directly to $f(f(x)) - x = 0$. Hence find a solution of this equation correct to 3 decimal places. Compare your answer to that obtained in (ii) (15 marks)

13. Let

$$f(x) = x^3 - (9/2)x^2 + 6x - (5/2).$$

Find intervals of x on which the function is (i) increasing, (ii) decreasing, (iii) concave up and (iv) concave down. Locate any zeros, extrema and inflection points. Classify the extrema. Sketch the graph. (15 marks)

14. A wall 4 feet high stands 32 feet from a building. Find the length of the shortest ladder that will reach the building from the ground outside the wall. (15 marks)

15. Obtain approximate values for the integral

$$\int_5^6 \frac{1}{x}\,dx$$

using (i) the trapezoidal rule and (ii) Simpson's rule with the interval [5, 6] subdivided into ten equal parts in each case. Give your answers correct to five decimal places. Compare your answers with the exact result and comment very briefly on your findings. (15 marks)

16. Given the formula for the surface area of a solid of revolution

$$S = 2\pi\int_a^b y ds = 2\pi\int_a^b y\sqrt{1+\left(dy/dx\right)^2}\,dx\ ,$$

show that if the curve $y = \sin x$ for $0 \le x \le \pi$ is rotated about the x-axis through 2π radians, then the area of the surface so formed is given by

$$S = 4\pi\int_0^1 \sqrt{1+u^2}\,du\ ,$$

where $u = \cos x$. Evaluate S with the further substitution $u = \sinh t$ and obtain

$$S = 2\pi(\sinh^{-1}(1) + \sqrt{2}).$$

(15 marks)