MATH101: JANUARY 2002 EXAMINATION

Instructions to candidates

Answer all of section A and THREE questions from section B. The marks shown against questions, or parts of questions, indicate their relative weights. Section A carries 55% of the total marks.

SECTION A

1. Write down the natural domain and the range of each of the following functions and sketch their graphs:

(i) $y = |x+2|$ (ii) $y = 1/(2x-1)^2$. (4 marks)

2. Find the general solution of

$$\sin\theta = \sqrt{3}/2 .$$ (4 marks)

3. If

$$f(x) = \frac{1+x}{1-2x} ,$$

find the corresponding inverse function $f^{-1}(x)$. Verify that

$$f(f^{-1}(x)) = x.$$ (5 marks)

4. Find the following limits

(i) $\lim_{x\to 2} \frac{x^4+x+5}{x+2}$ (ii) $\lim_{x\to -1} \frac{x^7+1}{x+1}$ (iii) $\lim_{x\to \pi} \frac{1+\cos x}{\tan^2 x}$. (6 marks)

5. Find

$$\frac{d}{dx}(2x^2 - x)$$

by differentiation from first principles. (4 marks)

6. Differentiate the following identifying any rules of differentiation that you use

(i) $y = x^3 e^{3x}$ (ii) $y = \frac{1+x}{1+x^2}$ (iii) $y = \sin(\tan(3x))$. (6 marks)

7. Use implicit differentiation to find the value of the derivative of

$$x^2y^2 + 3\sin x - 5y = -5$$

at the point (0,1). Find the equation of the tangent to the curve at this point. (6 marks)

8. Find and classify the local extrema of the function

$$f(x) = 2x^3 + \tfrac{1}{2}x^2 - x .$$ (5 marks)

9. Evaluate the following definite integrals

(i) $\int_0^1 x(x^2+1)^{10}\,dx$ (ii) $\int_0^{\pi/4} x\sin x\,dx$. (6 marks)

10. The curve C has equation $y = xe^x$, $0 \le x \le 1$. Find the volume of the solid of revolution generated when the finite region enclosed by C, the line $x = 1$ and the *x*-axis is rotated through 2π about the x-axis. (5 marks)

11. Use D'Alembert's ratio test to determine whether or not the following infinite series converges

$$\sum_0^\infty \frac{n^{100}}{2^n}.$$ (4 marks)

SECTION B

12. Let

$$f(x) = \frac{x^2+1}{x^2-4x+3}.$$

Find constants *A*, *B*, and *C* such that

$$f(x) = A + \frac{B}{x-1} + \frac{C}{x-3}.$$

Find intervals of *x* on which the function is (i) increasing, (ii) decreasing, (iii) concave up and (iv) convex up. Locate any zeros, asymptotes, extrema and inflection points. Classify the extrema. Sketch the graph. (15 marks)

13. Derive the formula

$$f(a) + f'(a)(x-a)$$

for the linear approximation to the function $f(x)$ near to $x = a$. Hence, or otherwise, show that, if x_0 is an approximation to a solution of the equation $f(x) = 0$, then

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

may give a better approximation.
By sketching graphs of $y = \sin 2x$ and $y = x$, demonstrate that the equation

$$\sin 2x - x = 0$$

has one and only one positive solution. Taking $x_0 = 1$, use the above method repeatedly to find an approximation to the above solution correct to 6 decimal places. (15 marks)

14 Write down the definitions of cosh$x$ and sinh$x$ in terms of e^x and e^{-x}. Hence find the values of x which satisfy the equation

$$7\cosh x - 3\sinh x = 7.$$

The hyperbolic tangent function is defined by $\tanh x = \sinh x / \cosh x$. Show that its inverse function $\tanh^{-1} x$ can be represented by the formula:

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right).$$

Use this formula to show that

(i) $\tanh^{-1}(x) + \tanh^{-1}(-x) = 0$ (ii) $\dfrac{d}{dx}\left(\tanh^{-1} x\right) = \dfrac{1}{1-x^2}$. (15 marks)

15 Obtain approximate values for the integral

$$\int_0^1 x\sqrt{1-x^2}\,dx$$

using (i) the trapezoidal rule and (ii) Simpson's rule with the interval [0,1] subdivided into eight equal parts in each case. Give your answers correct to five decimal places. Verify that (ii) is the more accurate method by evaluating the integral directly and comparing the results. (15 marks)

16 (i) Find the following indefinite integrals

(a) $\int e^{2x}\sin x\ dx$ (b) $\int \dfrac{x^3}{\sqrt{4-x^2}}\,dx,$ $(-2 < x < 2)$.

(ii) Find the arc length of the curve $y = x^{3/2}$ from $x = 1$ to $x = 2$ correct to 3 decimal places.

(15 marks)