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THE UNIVERSITY
of LIVERPOOL

JANUARY 2007 EXAMINATIONS

Bachelor of Arts: Year 1
Bachelor of Science : Year 1
Master of Mathematics : Year 1
Master of Physics : Year 1

FOUNDATION MODULE I

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

Answer all of Section A and THREE questions from Section B. The marks shown against questions, or parts of questions, indicate their relative weight. Section A carries 55% of the available marks.

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SECTION A

1. Determine the natural domain and range of each of the following functions and provide a rough sketch of its graph:

$$\text{(i)} \qquad y = \sqrt{x+1}, \qquad \text{(ii)} \qquad y = |1-x| + x\,.$$

[5 marks]

2. Find the general solution of

$$\cos\left(\theta + \frac{\pi}{4}\right) = -\frac{1}{2}\,.$$

[3 marks]

3. Find the inverse function $f^{-1}(x)$ of the function

$$f(x) = \frac{1-x}{3x-2} \qquad (x \neq \frac{2}{3})$$

and verify that

$$f(f^{-1}(x)) = x\,.$$

[5 marks]

4. Determine, giving reasons, whether the following functions are odd, even or neither:

$$\text{(i)} \qquad p(x) = \sin(x^2) + |x|\,, \qquad \text{(ii)} \qquad q(t) = \frac{t^3 - t + 1}{2t^2 + 5}\,.$$

[4 marks]

5. Find the following limits, where they exist:

$$\text{(i)} \lim_{x\to -2} \frac{x^2 - x - 6}{x+2}\,, \qquad \text{(ii)} \lim_{x\to 1} \frac{x^3 - 1}{\ln x}\,, \qquad \text{(iii)} \lim_{x\to \frac{\pi}{2}} \frac{\cos x}{1 + \cos(2x)}\,.$$

[6 marks]

6. Differentiate $3x^2 - x$ with respect to x from first principles, i.e. by using an appropriate limiting process. [4 marks]

7. Differentiate the following with respect to x, identifying any rules of differentiation that you use, and simplifying your answer where possible:

(i) $y = x^2 e^{3x}$, (ii) $y = \dfrac{1+x^2}{(x-2)^2}$, (iii) $y = \sqrt{1+\sin^2 x}$.

[6 marks]

8. Use implicit differentiation to find the slope of the curve given by

$$x^2y^2 + 3\sin x - 5y = -5$$

at the point $(0,1)$. Hence obtain the equation of the tangent to the curve at this point.

[6 marks]

9. Find and classify the local extrema of the function

$$g(x) = 12x^3 - 3x^2 - 6x - 3\,.$$

[5 marks]

10. Evaluate the following integrals:

(i) $\displaystyle\int x\sqrt{1+x^2}\,dx$, (ii) $\displaystyle\int_0^{\frac{\pi}{4}} x\sin x\,dx$.

[6 marks]

11. Use D'Alembert's ratio test to determine whether or not the following infinite series converges

$$\sum_{n=1}^{\infty} \frac{2^n+5}{3^n}\,.$$

[5 marks]

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SECTION B

12. (i) Given that $x = c$ satisfies the equation $f(x) = x$, verify that $x = c$ also satisfies the equation $f(f(x)) = x$.

(ii) The function f is defined by

$$f(x) = x(x-2)\,.$$

Find the solutions of the equation

$$f(x) = x\,.$$

(iii) Show that $f(f(x)) = x^4 - 4x^3 + 2x^2 + 4x$ and use the result of (i) to find *all* the solutions of the equation

$$f(f(x)) = x\,.$$

(iv) Apply the Newton-Raphson method with initial value $x_0 = 3/2$ directly to $f(f(x)) - x = 0$ to find a solution of this equation correct to 3 significant figures. Compare your answer to that obtained in part (iii).

[15 marks]

13. Given that

$$f(x) = \frac{x^2 + 2x + 1}{x - 1}\,,$$

find constants A, B and C such that

$$f(x) = A + Bx + \frac{C}{x-1}\,.$$

Find intervals of x on which the function is (i) increasing (ii) decreasing (iii) concave up and (iv) concave down.

Locate any zeros, asymptotes, extrema and inflection points. Classify the extrema and then sketch the graph.

14. Anne is at point A on the bank of a long straight stretch of still water of width 1/5 km. She enters the water directly and can swim at 4 km/hour. She wishes to reach the hot showers at point B on the opposite bank. If she swims straight across, she will then have to walk a distance of 1km to reach B. Suppose that she can walk/jog at 12km/hour. To which point C on the opposite bank should she swim (in a straight line) so as to get to the hot shower as soon as posible?

    How long will it take her to get to the shower at B? (Give your answer to the nearest minute.)

    [15 marks]

15. Obtain approximate values for the integral

$$\int_1^2 \frac{2}{x+3}\,dx$$

    using (i) the trapezoidal rule and (ii) Simpsons' rule with the interval $[1,2]$ divided into 10 equal sub-intervals in each case. Give your answers correct to 5 decimal places.

    Compare your answer with the exact result and comment very briefly on your findings.

    [15 marks]

16. (i) Find the following indefinite integrals

$$\text{(a)} \quad \int e^{2x}\cos x\,dx\,, \qquad \text{(b)} \quad \int \frac{2x^3}{\sqrt{4-x^2}}\,dx\,, \qquad (-2 < x < 2)\,.$$

    (ii) Find the arc length of the curve $y = x^{3/2}$ from $x = 1$ to $x = 2$, correct to 3 decimal places.

    [15 marks]