M101

M101 EXAMINATION JAN 2001

Instructions to candidates

Answer all of section A and THREE questions from section B. The marks shown against questions, or parts of questions, indicate their relative weights. The total of marks available in section A is 55.

SECTION A

1. Sketch graph of $y = |\sin x|$ for $-\pi \leq x \leq \pi$. [3 marks]

2. Find the general solution (for real θ) of the equation

$$\cos\theta = \frac{1}{2}.$$

[3 marks]

3. Find the domain of x such that x satisfies the inequality

$$|x+2| < 3.$$

[2 marks]

4. If

$$f(x) = \frac{x}{1+2x},$$

find the corresponding inverse function $f^{-1}(x)$. Verify that

$$f\left(f^{-1}(x)\right) = x.$$

[5 marks]

5. Which of the following limits exist? Determine the values of those that do.

$$(a)\quad \lim_{x\to 0}\frac{\tan x}{x} \qquad (b)\quad \lim_{x\to -1}\frac{x^5+1}{x+1} \qquad (c)\quad \lim_{x\to 2}\frac{x^2+5x+4}{x-2}.$$

[6 marks]

6. Differentiate the following functions with respect to x:

$$(a)\quad e^{x^2} \qquad (b)\quad x^3\sin x \qquad (c)\quad \frac{x^2}{\ln x}.$$

[7 marks]

7. Find and classify the local extrema of the function

$$f(x) = x^3 - 6x.$$

[6 marks]

8. Define $\cosh x$ and $\sinh x$ in terms of e^x and e^{-x}. From the definitions, prove that

$$\cosh^2 x - \sinh^2 x = 1.$$

[4 marks]

9. Find the equation of the tangent to the curve

$$x^2 + y^2 = 5(y - x)$$

at the point $(x, y) = (1, 2)$. [6 marks]

10. Evaluate the following integrals:

$$(a)\quad \int_0^{\frac{\pi}{6}} \sin 3x \, dx \qquad (b)\quad \int_0^1 \frac{x}{(1+x^2)} \, dx \qquad (c)\quad \int_1^2 \ln x \, dx.$$

[9 marks]

11. The curve C has equation $y = x^4$, $\quad 0 \le x \le 1$. Find the volume of the solid of revolution generated when the finite region enclosed by the curve C, the line $x = 1$ and the x-axis is rotated through 2π about the x-axis.

[4 marks]

SECTION B

12. Let

$$f(x) = \frac{x^2 + x + 1}{(x+2)(x-1)}.$$

(i) Find constants A, B, C such that

$$f(x) \equiv A + \frac{B}{x+2} + \frac{C}{x-1}.$$

(ii) Prove that f has only one stationary point, and determine its nature. Does f have any points of inflection ?

(iii) Sketch the graph $y = f(x)$, indicating clearly the positions of asymptotes and the stationary point.

13. Suppose x_0 is an approximation to a solution of the equation $f(x)=0$. By considering the tangent to the graph of $y = f(x)$ at $x = x_0$, or otherwise, explain why x_1 may give a better approximation, where x_1 is given by the Newton-Raphson formula:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

By sketching the graphs of $f(x) = \ln x$ and $f(x) = \frac{1}{x}$, demonstrate that the equation

$$x \ln x - 1 = 0 \qquad (13.1)$$

has one and only one solution. Choosing an appropriate approximate solution x_0 to Eq. (13.1), use the Newton-Raphson method to find x_1. Use x_1 in turn to find another approximation x_2. Test whether x_2 is in fact a better approximation to the exact result than x_0.

[15 marks]

14. (a) Find the arc length of the curve $y = \frac{4}{3}x^{\frac{3}{2}}$ from $x = 0$ to $x = 2$.

(b) Sketch the curve $y = \tan x$ from $x = 0$ to $x = \frac{\pi}{2}$. Find the area of the finite region R bounded by the curve $y = \tan x$, the line $x = \frac{\pi}{4}$ and the x-axis. Calculate also the volume of the solid of revolution generated when the region R is rotated through 2π radians about the x-axis.

[15 marks]

15. (i) Starting from $\frac{d}{dx}\sin x = \cos x$ and $\frac{d}{dx}\cos x = -\sin x$, show that

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \text{and} \qquad \frac{d}{dx}\sec x = \sec x \tan x.$$

(ii) Show that

$$\int_0^{\frac{\pi}{4}} \sec^2\theta \, d\theta = 1.$$

(iii) Defining $I_n = \int_0^{\frac{\pi}{4}} \sec^n\theta \, d\theta$, show by writing $\sec^{n+2}\theta = \sec^n\theta . \sec^2\theta$ and integrating by parts, that

$$(n+1)I_{n+2} = nI_n + (\sqrt{2})^n.$$

Verify that this formula reproduces the result of (*ii*) above, and evaluate

$$I_6 = \int_0^{\frac{\pi}{4}} \sec^6 x \, dx.$$

[15 marks]

16. S_n is defined as follows:

$$S_n = 1 + x + \cdots + x^{n-1} = \sum_{r=0}^{n-1} x^r.$$

By writing down xS_n, or otherwise, show that if $x \neq 1$ then

$$S_n = \frac{1-x^n}{1-x}.$$

Hence justify the statement that the geometric series S_∞ converges to $(1-x)^{-1}$ for $|x| < 1$, i.e.

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots \quad \text{for} \quad |x| < 1. \tag{16.1}$$

Show that the ratio test also leads to the conclusion that this series converges for $|x| < 1$.

By integrating both sides of the Eq. (16.1), obtain a series for $\ln(1-x)$. Give a reason why you would expect this series to also converge for $|x| < 1$. Verify your conclusion by once again using the ratio test. Discuss briefly the cases $x = \pm 1$.

[15 marks]