SECTION A

1. Given that $z = 3 + 2i$, write down the complex conjugate $\bar{z}$.

Hence show that $\frac{\bar{z}}{z} + \frac{z}{\bar{z}} = \frac{10}{13}$. [4 marks]

2. Express the complex number $1 + i$ in polar form. Hence, by the use of De Moivre's theorem or otherwise, evaluate $(1 + i)^6$.
Draw a diagram of the complex plane identifying <u>clearly</u> all the points corresponding to the complex numbers z which satisfy the equation

$$z^4 = 1 + i.$$ [7 marks]

3. Sketch the graph of $y = 7 + \frac{3}{x}$. Hence or otherwise find the range(s) of values of x which satisfy the inequality: $\left| 7 + \frac{3}{x} \right| > 10.$ [6 marks]

4. Find $\frac{d}{dx}\left(3x^2 - 4\right)$ by differentiation <u>from first principles using a limiting process</u>. [4 marks]

5. Evaluate each of the following limits by any valid method:

(i) $\lim_{x \to \infty} \frac{3x^3 + 4}{(x+1)(2x+1)(3x+1)}$, (ii) $\lim_{x \to 1} \frac{7x^3 - 17x^2 + 13x - 3}{3x^3 - 7x^2 + 5x - 1}$. [6 marks]

6. Find the derivatives of the following functions with respect to x <u>using the rules of differentiation.</u> Simplify your answers as far as possible.

(i) $(1 + x^3)\ln(1 + x^3)$, (ii) $\frac{\cos x}{1 + \sin x}$. [6 marks]

7. Evaluate the following definite integrals:

$$(i)\ \int_0^1 x^2 e^x\, dx\,, \qquad (ii)\ \int_2^3 \frac{dx}{x(1-x)}\,.$$ [7 marks]

8. Given that $x^3 + 3xy + y^3 = 5$, where y is a function of x,

find the value of $\dfrac{dy}{dx}$ when $x = y = 1$. [3 marks]

9. Draw a clear diagram showing the region A enclosed between the y-axis and the curves $y = x^2$, $y = 2 - x^2$ for $x \geq 0$.

Deduce that the area of A is 4/3.

The region A is now rotated through 360° about the x-axis. Show that the volume of the solid generated is $8\pi/3$. [7 marks]

10. Use D'Alembert's ratio test to determine whether or not the following infinite series converges.

$$\sum_{n=1}^{\infty} \frac{5^n (n+1)!(n+2)!(n+3)!}{(3n)!}.$$ [5 marks]

SECTION B

11. Write down the binomial expansion for $(\cos\theta + i\sin\theta)^5$.

Use De Moivre's theorem to deduce that:

$$\cos(5\theta) = (16\cos^4\theta - 20\cos^2\theta + 5)\cos\theta, \quad \text{and}$$

$$\sin(5\theta) = (16\cos^4\theta - 12\cos^2\theta + 1)\sin\theta.$$

Hence or otherwise obtain an expression for $\cos(6\theta)$ in terms of powers of $\cos\theta$.

Check the validity of your result by substituting $\theta = \pi/3$.

[15 marks]

12. (i) The function f is defined by $f(x) = \dfrac{\alpha x + 3}{4x - 7}$ for $x \neq 7/4$, where α is a constant parameter. Find $f^{-1}(x)$, where f^{-1} is the inverse function of f.

Hence determine the value of α which makes $f^{-1}(x) = f(x)$ for all values of $x \neq 7/4$. [5 marks]

(ii) Given that $x = a$ satisfies the equation $f(x) = x$, verify that $x = a$ also satisfies the equation $f[f(x)] = x$.

The function f is defined by $f(x) = x(x-3)$. Obtain the solutions of the equation $f(x) = x$. Verify that $f[f(x)] = x^4 - 6x^3 + 6x^2 + 9x$.

Hence find all <u>four</u> values of x which satisfy the equation $f[f(x)] = x$.

[10 marks]

13. Use the substitution $u = x^2$ to find the real values of x which satisfy the equation $x^4 - 3x^2 - 4 = 0$. Hence determine the stationary points of the function f defined by: $f(x) = x^5 - 5x^3 - 20x + 38$.

Find $f''(x)$ and deduce how the <u>concavity</u> of the graph $y = f(x)$ will change as x increases from large negative to large positive values.

Now draw the graph of $y = f(x)$ <u>carefully</u>, identifying turning points and all the points of inflexion. <u>Note</u> that for full credit all mathematical working must be shown in detail.

Verify from your graph that the equation $f(x) = 0$ has a root $x = \alpha$ where $1 < \alpha < 2$. Use the Newton-Raphson method to evaluate α: starting with $x_0 = 1.0$, compute x_n where n is sufficiently large for α to be correct to <u>seven</u> decimal places. State $f(x_n)$ to illustrate the accuracy of your result.

[15 marks]

14. Obtain approximate values for the integral: $\int_0^1 x\sin(\pi x^2)\,dx$ using

(i) the trapezoidal rule and (ii) Simpson's rule with the interval $[0,1]$ subdivided into ten equal parts in each case, working throughout with <u>at least seven</u> significant digits. Verify that (ii) is the more accurate method by evaluating the integral directly and comparing the results.

[15 marks]

15. (i) Write down the definition of $\cosh x$ in terms of e^x and e^{-x}.

Verify that $2\cosh^2 x = 1 + \cosh(2x)$ and hence obtain an expression for $\cosh^4 x$ involving $\cosh(2x)$ and $\cosh(4x)$. Deduce that:

$$32\int_0^1 \cosh^4 x\,dx = 12 + 8\sinh(2) + \sinh(4).$$

[6 marks]

(ii) The firm Gasflo Plc. (motto: "No Gassle!") has a pumping station S situated on the edge of a circular lake of radius r with its centre at the point O. The firm's managing director Sid would like gas to be piped from S to his luxury mansion, which is on the other side of the lake at a point B diametrically opposite S.

The plan is to lay a straight pipe underwater from S to a point A on the edge of the lake, and then on around the edge to B with a pipe buried in a trench on land.

Assuming that the cost per unit length of laying a pipe underwater is twice that on land, show that the total cost incurred is proportional to:

$C(\theta) = 4r\cos\theta + 2r\theta$, where the angle $AOB = 2\theta$ and $0 \le \theta \le \pi/2$.

Hence, by sketching the graph of $C(\theta)$ or otherwise, find the route chosen by Gasflo to minimize the cost of supplying gas to Sid's mansion.

[9 marks]

MATH101: JANUARY 2000 EXAMINATION

You may attempt all questions.
All answers to Section A and the best THREE answers
from Section B will be taken into account.
The marks shown indicate the relative weights of the questions.
Section A carries 55% of the available marks.