PAPER CODE NO.
MATH 014

THE UNIVERSITY
of LIVERPOOL

MAY EXAMINATIONS 2007

Bachelor of Engineering : Foundation Year
Bachelor of Engineering : Year 1
Bachelor of Science : Foundation Year
Bachelor of Science : Year 1
No qualification aimed for: Year 1

DIFFERENTIAL EQUATIONS AND APPLICATIONS TO MECHANICS

TIME ALLOWED : Three Hours

INSTRUCTIONS TO CANDIDATES

ALL answers to Section A and the best THREE answers to Section B will be counted.
Section A carries 55% of the available mark.
The marks shown against sections of questions indicate their relative weights.
Your attention is drawn to the Formulae Sheet which accompanies this exam paper.

SECTION A

1. Evaluate the following indefinite integrals:

$$\text{(i)} \int \tfrac{1}{3}x^4\,dx \quad [2\text{ marks}]\ , \qquad \text{(ii)} \int \sqrt{7x+4}\,dx \quad [2\text{ marks}]\ ,$$

$$\text{(iii)} \int \sin(2x+4)\,dx \quad [3\text{ marks}]\ , \qquad \text{(iv)} \int e^{-\frac{2}{3}x}\,dx \quad [3\text{ marks}]\ .$$

2. Evaluate the following definite integrals:

$$\text{(i)} \int_1^3 \frac{dx}{4x-3} \quad [2\text{ marks}]\ , \qquad \text{(ii)} \int_0^2 3x(x^2-2)\,dx\ [3\text{ marks}]\ ,$$

$$\text{(iii)} \int_0^{2\pi} \sin(\tfrac{1}{2}x)\,dx \quad [2\text{ marks}]\ , \qquad \text{(iv)} \int_{1/3}^{2/3} (3x-1)^7\,dx\ [3\text{ marks}]\ .$$

3. Using partial fractions, the following rational functions can be written as

$$\text{(i)} \quad \frac{-2x-34}{(x+5)(x-3)} = \frac{A}{x+5} + \frac{B}{x-3}\ ,$$

$$\text{(ii)} \quad \frac{-x^2-3x+2}{(x+1)(x^2+1)} = \frac{C}{x+1} + \frac{Dx+E}{x^2+1}\ .$$

Compute the constants A, B, C, D, E. [5 marks]

Hence evaluate the following integrals:

$$\text{(iii)} \quad \int \frac{-2x-34}{(x+5)(x-3)}\,dx\ ,$$

[2 marks]

$$\text{(iv)} \quad \int_0^2 \frac{-x^2-3x+2}{(x+1)(x^2+1)}\,dx\ .$$

[3 marks]

4. Use integration by parts to show that

$$\int_0^{2/7} 7xe^{-7x}\,dx = \frac{1}{7}\left(1-\frac{3}{e^2}\right) .$$

[7 marks]

5. Find the general solutions of the following first order differential equations. *Hint:* try to use separation of variables.

$$\text{(i)}\ \ \frac{dy}{dx} = \sin(3x) ,$$

[3 marks]

$$\text{(ii)}\ \ \frac{dy}{dx} = 3y \quad (\text{where }\ y>0) .$$

[3 marks]

For equation (ii) find the particular solution where $y=1$ when $x=1$. [2 marks]

6. Find the general solutions of the following second order differential equations

$$\text{(i)}\ \ 3\frac{d^2y}{dx^2} + 9\frac{dy}{dx} - 30y = 0 ,$$

[4 marks]

$$\text{(ii)}\ \ \frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0 .$$

[4 marks]

For equation (ii), find the particular solution where $y=4$ and $\dfrac{dy}{dx}=2$ when $x=0$. [2 marks]

SECTION B

7. The driver of a car is very careful about speed cameras and never exceeds the speed limit, but always travels as fast as he can. He is travelling on roads with a speed limit of 54 kilometres per hour ($= 15\,ms^{-1}$). His car can accelerate at a maximum rate of $3\,ms^{-2}$ and the brakes can decelerate it at a maximum rate of $5\,ms^{-2}$.

(a) What is the shortest distance, in metres, in which he can accelerate from rest to a speed of 54 kilometres per hour? [3 marks]

(b) What is the shortest distance, in metres, in which he can stop from a speed of 54 kilometres per hour? [3 marks]

(c) What is the shortest time, in seconds, in which he can travel from rest at his house to rest in the car park at his local railway station given that the station is $1\,km$ from his house? [4 marks]

(d) It takes 250 seconds to travel the $4\,km$ from the station to his work on the train. He arrives at work right on time. If he decides to drive all the way from his house to work instead of getting the train, leaving his house at the same time, can he still get to work on time? (*Hint:* calculate the shortest time in which he can travel the $5\,km$ total distance from his house to work in his car and compare with the total time to drive from his house to the station and to get from the station to work on the train.) [5 marks]

8. (a) Find the general solution of the second order differential equation

$$4\frac{d^2y}{dx^2} + 20\frac{dy}{dx} + 25y = 0$$

and find the particular solution where $y = 0$ and $\frac{dy}{dx} = 1$ when $x = 2$.

[8 marks]

(b) Evaluate the definite integral

$$\int_0^{\sqrt{3}} \frac{dx}{\sqrt{3 - x^2}}$$

(*Hint:* You may use the substitution $x = \sqrt{3}\sin(u)$.) [7 marks]

9. A cricket ball, modelled as a particle of mass M that experiences no air-resistance, is thrown vertically upwards from the ground with an initial speed of $25\,ms^{-1}$.

(a) Using the notation that the vertically upwards direction is denoted by y, with the origin, $y = 0$, at ground level, show that the differential equation governing the motion of the ball is

$$\frac{d^2y}{dt^2} = -g,$$

where g is the acceleration due to gravity and t is time with $t = 0$ at the moment the ball is thrown. [3 marks]

(b) Find the particular solution of this differential equation that satisfies the given initial conditions. [4 marks]

(c) Hence find the maximum height reached by the ball, using $g = 10\,ms^{-2}$. [4 marks]

(d) Find the height reached by the ball on its way up from the ground when its speed has reduced to $10\,ms^{-1}$. [4 marks]

10. A simple pendulum makes an angle θ with the vertical. Its motion is approximately described by the differential equation

$$\frac{d^2\theta}{dt^2} + k^2\theta = 0$$

where t is time and $k^2 = 36$.

(a) Given that the initial conditions are $\theta = 1$ and $\dfrac{d\theta}{dt} = 6$ at $t = 0$, solve this differential equation. [5 marks]

(b) Show by substitution that

$$\theta(t) = \sqrt{2}\sin(6t + \tfrac{\pi}{4})$$

is a solution of the differential equation and satisfies the initial conditions.

[5 marks]

(c) Plot this function on a graph, where the horizontal axis is t and the vertical axis is θ, such that you display at least one full period of the function. Specify explicitly the period of the function. [5 marks]

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MATH 014 Formulae Sheet

Trigonometric Identities

$$\tan x = \frac{\sin x}{\cos x}\ , \quad \cot x = \frac{\cos x}{\sin x}\ , \quad \sec x = \frac{1}{\cos x}\ , \quad \operatorname{cosec} x = \frac{1}{\sin x}.$$

$$\sin^2 x + \cos^2 x = 1\ , \quad \tan^2 x + 1 = \frac{1}{\cos^2 x} = \sec^2 x\ .$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y\ ,$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y\ ,$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}\ ,$$

$$\sin(2x) = 2 \sin x \cos x\ ,$$

$$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x\ ,$$

$$\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos x}{2}}\ , \quad \cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1+\cos x}{2}}\ .$$

Differentiation Rules

Chain rule:

$$\frac{d}{dx} f[u(x)] = \frac{df}{du}\frac{du}{dx}\ .$$

Product rule:

$$\frac{d}{dx}[u(x)v(x)] = \frac{du}{dx}v(x) + u(x)\frac{dv}{dx}\ .$$

Quotient rule:

$$\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) = \frac{1}{v^2(x)}\left(\frac{du}{dx}v(x) - u(x)\frac{dv}{dx}\right)\ .$$

List of Derivatives

$f(x)$:	cx^n	$\frac{c}{x}$	$\frac{c}{x^n}$	ce^x	$c\ln x$	$\sin x$	$\cos x$	$\tan x$
$\frac{df}{dx}$:	$c\,nx^{n-1}$	$-\frac{c}{x^2}$	$-\frac{nc}{x^{n+1}}$	ce^x	$\frac{c}{x}$	$\cos x$	$-\sin x$	$\frac{1}{\cos^2 x} = \sec^2 x$

List of Integrals (all $+\,C$)

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
$(ax+b)^n$	$\frac{(ax+b)^{n+1}}{a(n+1)}$, $(n \neq -1)$	$\frac{1}{ax+b}$	$\frac{1}{a}\ln\lvert ax+b\rvert$
$\frac{1}{(ax+b)^2}$	$-\frac{1}{a(ax+b)}$	$\frac{1}{(ax+b)(cx+d)}$	$\frac{1}{ad-bc}\ln\left\lvert\frac{ax+b}{cx+d}\right\rvert$, $(ad-bc\neq 0)$
$\frac{1}{ax^2+b}$	$\frac{1}{\sqrt{ab}}\arctan\left(\sqrt{\frac{a}{b}}\,x\right)$, $(a,b>0)$	$\frac{1}{ax^2-b}$	$\frac{1}{2\sqrt{ab}}\ln\left\lvert\frac{\sqrt{a}x-\sqrt{b}}{\sqrt{a}x+\sqrt{b}}\right\rvert$, $(a,b>0)$
$\frac{x}{ax^2+b}$	$\frac{1}{2a}\ln\lvert ax^2+b\rvert$	e^{ax}	$\frac{1}{a}e^{ax}$
$\sin(ax)$	$-\frac{1}{a}\cos(ax)$	$\cos(ax)$	$\frac{1}{a}\sin(ax)$

Substitution

$$\int f(x)\,dx = \int \left(h(u)\frac{dx}{du}\right)\,du\,, \quad \text{where} \quad f(x) = h(u(x))\,.$$

$$\int_a^b f(x)dx = \int_{u(a)}^{u(b)} \left(h(u)\frac{dx}{du}\right)\,du\,, \quad \text{where} \quad f(x) = h(u(x))\,.$$

Integration by parts

$$\int u(x)\frac{dv}{dx}\,dx = u(x)v(x) - \int \frac{du}{dx}v(x)\,dx + C\,.$$

$$\int_a^b u(x)\frac{dv}{dx}\,dx = [u(x)v(x)]_a^b - \int_a^b \frac{du}{dx}v(x)\,dx\,.$$

Differential Equations

Separation of Variables

Equation	Solution
$\frac{dy}{dx} = f(x)$	$y = \int f(x)dx$
$\frac{dy}{dx} = ay$	$y = e^{ax+C}$, for $y > 0$.

Second order, linear, unforced (homogeneous) differential equations

Standard form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0\,. \tag{1}$$

Characteristic equation:

$$a\alpha^2 + b\alpha + c = 0\,. \tag{2}$$

Solution of characteristic equation (2) for $a \neq 0$:

$$\alpha_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Discriminant: $D = b^2 - 4ac$.

General solution of the differential equation (1):

1. Case I: $D > 0$ ($a \neq 0$).
   Characteristic equation (2) has two distinct real solutions $\alpha_{1,2}$.
   General solution of (1):

$$y(x) = A_1 e^{\alpha_1 x} + A_2 e^{\alpha_2 x} .$$

2. Case II: $D = 0$ ($a \neq 0$).
   Characteristic equation (2) has one real solution $\alpha = \alpha_1 = \alpha_2 = -\frac{b}{2a}$.
   General solution of (1):

$$y(x) = (A_1 + A_2 x)e^{\alpha x} .$$

3. Case III: $D < 0$ ($a \neq 0$).
   Characteristic equation (2) has no real solutions.
   General solution of (1) for the case $b = 0$:

$$y(x) = A_1 \sin(\omega x) + A_2 \cos(\omega x) , \quad \text{where} \quad \omega = \sqrt{\frac{c}{a}} .$$

4. Degenerate case: $a = 0$. (First order equation.)
   Characteristic equation (2) has solution $\alpha = -\frac{c}{b}$.
   General solution of (1):

$$y(x) = Ae^{\alpha x} .$$