SECTION A

1. Evaluate the following indefinite integrals

(i) $\int \frac{3}{2}x^5\,dx$ [2 marks] (ii) $\int \sqrt{3x-7}\,dx$ [2 marks]

(iii) $\int \cos(4x+2)\,dx$ [3 marks] (iv) $\int e^{-5x}\,dx$ [3 marks]

2. Evaluate the following definite integrals

(i) $\int_2^3 \frac{dx}{2x-3}$ [3 marks] (ii) $\int_0^1 x(1-3x^2)\,dx$ [2 marks]

(iii) $\int_{\pi/5}^{2\pi} \sin(5x)\,dx$ [3 marks] (iv) $\int_0^2 (-x+1)^6\,dx$ [2 marks]

3. Using partial fractions, the following rational functions can be written as

(a) $$\frac{2}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$$

(b) $$\frac{x^2-1}{x(x^2+1)} = \frac{C}{x} + \frac{Dx+E}{x^2+1}$$

Compute the constants A, B, C, D, E. [5 marks]

Hence evaluate the following integrals:

(i) $\int \frac{2}{(x-3)(x+1)}\,dx$ [2 marks]

(ii) $\int_1^2 \frac{x^2-1}{x(x^2+1)}\,dx$ [3 marks]

4. Use integration by parts to show that

$$\int_0^{\pi/2} x\sin(\tfrac{x}{2})dx = (4-\pi)\tfrac{1}{2}\sqrt{2}$$

[7 marks]

5. Solve the following first order differential equations.
Hint: try to use separation of variables.

$$\text{(i)} \quad \frac{dy}{dx} = 3x^2 \quad \text{[3 marks]}$$

$$\text{(ii)} \quad \frac{dy}{dx} = 4y^2 \ \text{[3 marks]}$$

For equation (ii) find the particular solution where $y = 1$ when $x = \frac{1}{2}$. [2 marks]

6. Solve the following second order differential equations

$$\text{(i)} \quad \frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 8y = 0 \quad \text{[4 marks]}$$

$$\text{(ii)} \quad \frac{d^2y}{dx^2} + 36y = 0 \quad \text{[4 marks]}$$

For equation (ii), find the particular solution where $y = 1$ and $\frac{dy}{dx} = 1$ when $x = 0$.
[2 marks]

SECTION B

7. A ship of mass M looses power while sailing with initial speed v_0. Assume that the resisting force of the water is proportional to the speed v of the ship.

Show that the velocity of the ship is governed by the differential equation

$$M\frac{dv}{dt} = -kv$$

where t is time, $v = v(t)$ is the speed of the ship at time t, and $k > 0$ is a positive constant. [4 marks]

Solve the above differential equation and find the particular solution where the velocity is v_0 at time $t = 0$. [4 marks]

Given that the mass of the ship is $M = 65,000,000\,kg$, that the initial velocity is $v_0 = 8\frac{m}{s}$, and given that $k = 5,000\frac{kg}{s}$, how long will it take for the speed of the ship to drop to $v_1 = 1\frac{m}{s}$? [4 marks]

How far will the ship travel before reaching this speed? [3 marks]

8. (i) Solve the following second order differential equation

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$$

and find the particular solution where $y = e^{-4}$ and $\frac{dy}{dx} = e^{-4}$ when $x = 1$.

[8 marks]

(ii) Evaluate the following integral

$$\int_0^{\pi/4} \sin^3(2x)\cos(2x)dx$$

Hint: You may use the substitution $u = \sin(2x)$. [7 marks]

9. A train starts at rest at point A. With uniform acceleration $0.25\frac{m}{s^2}$ it reaches its maximal speed v_{max} at point B after 2 minutes.

Show that its maximal speed is $v_{max} = 30\frac{m}{s}$. [3 marks]

Express v_{max} in $\frac{km}{h}$. [1 marks]

Compute the distance between points A and B. [2 marks]

After passing point B, the train proceeds with maximal speed $v_{max} = 30\frac{m}{s}$ for 10 minutes until it reaches point C. What is the distance between points B and C? [2 marks]

From point C the train decelerates uniformly with $0.6\frac{m}{s^2}$ until it stops at point D. How long does it take the train to get from C to D? [3 marks]

What is the distance between C and D, and, hence, what is the total distance the train has travelled? [4 marks]

10. A simple pendulum makes an angle of θ with the vertical. Its motion is approximately described by the differential equation

$$\frac{d^2\theta}{dt^2} + k^2\theta = 0$$

where t is time and $k^2 = 25$. Given that $\theta = 1$ and $\frac{d\theta}{dt} = 5$ at $t = 0$, solve the above differential equation. [5 marks]

Show by substitution that

$$\theta(t) = \sqrt{2}\sin(5\,t + \tfrac{\pi}{4})$$

solves both the above differential equation and the initial condition. [5 marks]

Plot this function on a graph, where the horizontal axis is t and the vertical axis is θ, such that you display at least one full period of the function. Specify explicitly the period of the function. [5 marks]