SUMMER 1999 EXAMINATIONS

DIFFERENTIAL EQUATIONS AND APPLICATIONS TO MECHANICS

TIME ALLOWED : Three Hours

INSTRUCTIONS TO CANDIDATES

All answers to Section A and the best THREE answers to Section B will be counted. Section A carries 55% of the available marks. The marks shown against sections of questions indicate their relative weights.

In this paper bold-face quantities like $\mathbf{F}$ denote vectors, and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ denote unit vectors in the x, y, z directions respectively.

SECTION A

1. Find the values of x which satisfy the equation

$$\frac{x^2+1}{2x^2-3x+3} - 1 = 0.$$

[3 marks]

2. Find the derivatives of the functions

(*a*) $(x-1)(x^2+2)$, [2 marks]

(*b*) $\cos^2 x$, [2 marks]

(*c*) e^{2x}, [2 marks]

(*d*) $\frac{x^2+1}{2x^2-3x+3}$. [3 marks]

3. Given that $y = \sin x - x\cos x$, calculate dy/dx and d^2y/dx^2. Show that

$$\frac{d^2y}{dx^2} + y = 2\sin x.$$

[4 marks]

4. Evaluate the integral

$$\int_0^1 (x^2+1)^2\,dx.$$

[3 marks]

5. Integrate by parts

$$\int_1^2 xe^x\,dx.$$

[4 marks]

6. Use the substitution $u = x^2$ to evaluate the following integral:

$$\int_0^{\sqrt{\pi}} x\sin x^2\,dx.$$

[5 marks]

7. Express

$$\frac{x}{(x+2)(x+3)}$$

in partial fractions. Hence evaluate

$$\int \frac{x}{(x+2)(x+3)}\,dx.$$

[5 marks]

8. Use the method of separation of variables to find the general solution of the differential equation

$$x^2 \frac{dy}{dx} = y^2.$$

[6 marks]

9. Find an integrating factor for the differential equation

$$\frac{dy}{dx} + 2y = 1.$$

Hence find the general solution.

Find the particular solution which takes the value $y = \frac{1}{2}$ at $x = 0$.

[6 marks]

10. Find the general solutions of the following second order differential equations:

$$(a) \quad \frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 5y = 0,$$

[3 marks]

$$(b) \quad \frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0,$$

[3 marks]

$$(c) \quad \frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 13y = 0.$$

[4 marks]

SECTION B

11. A law-abiding motorcyclist has a maximum acceleration of 3 ms^{-2} and a maximum deceleration of 4 ms^{-2}. He never exceeds the speed limit of 30 miles per hour (13.4 ms^{-1}).

What is the shortest distance in which he can accelerate from rest to 30 mph?

What is the shortest distance in which he can stop from 30 mph?

What is the shortest time in which he can travel from rest at A to rest at B given that the distance AB is 50 m?

What is this shortest time in the case that AB is 100 m? [15 marks]

12. At time $t = 0$ a particle of mass 3 kg is at the origin and its velocity is $2\mathbf{i}\,\text{ms}^{-1}$. A constant force $\mathbf{F}$ is applied for 6 seconds, after which the particle has a velocity $2\mathbf{j}\,\text{ms}^{-1}$.

Find the force $\mathbf{F}$ and the position vector of the particle at time $t = 6$s.

[15 marks]

13. Find a particular integral of the form $ax^2 + bx + c$ for the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 3x^2.$$

Check that the particular integral which you have obtained does in fact satisfy the differential equation. Obtain the general solution of this differential equation. Find the particular solution which satisfies the boundary conditions

$$y = \frac{dy}{dx} = 0 \quad \text{at} \quad x = 0.$$

[15 marks]

14. A particle of mass m is released from rest at time $t = 0$ and falls under gravity in a medium which exerts a resistive force on the particle proportional to the speed. Let x be the distance through which it has fallen at time t and let v be its speed then. Show that

$$\frac{dv}{dt} = g - kv$$

where k is a positive constant. Integrate this equation by separating the variables to show that

$$v = \frac{g}{k}(1 - e^{-kt}).$$

Write $v = \frac{dx}{dt}$ and integrate again to find the distance x fallen in time t.

[15 marks]