SUMMER 1998 EXAMINATIONS

DIFFERENTIAL EQUATIONS AND APPLICATIONS TO MECHANICS

TIME ALLOWED : Three Hours

INSTRUCTIONS TO CANDIDATES

All answers to Section A and the best THREE answers to Section B will be counted. Section A carries 55% of the available marks. The marks shown against sections of questions indicate their relative weights.

In this paper bold-face quantities like $\mathbf{F}$ denote vectors, and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ denote unit vectors in the x, y, z directions respectively.

SECTION A

1. Find the derivatives of the functions

$$(a) \qquad (1-x)^3,$$

$$(b) \qquad \frac{x+2}{2x+3},$$

$$(c) \qquad e^x \sin x.$$

[7 marks]

2. Given the identity

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

obtain a formula for $\cos 2A$ in terms of $\sin A$. [2 marks]

Hence or otherwise evaluate the integral

$$\int_{\pi/4}^{\pi/2} \sin^2 x \, dx.$$

[4 marks]

3. Use the substitution $y = \sin x$ to evaluate the integral

$$\int_0^{\pi/4} \sin^3 x \cos x \, dx.$$

[4 marks]

4. Evaluate the following integral by parts:

$$\int_0^1 x \, e^x \, dx.$$

[3 marks]

5. Obtain an integrating factor for the differential equation

$$\frac{dy}{dx} - \frac{2y}{x} = 2x^3.$$

Hence find the general solution for the differential equation. [5 marks]

6. Find the general solutions of the differential equations

$$(a) \quad \frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0,$$

$$(b) \quad \frac{d^2y}{dx^2} - 2\frac{dy}{dx} = -y,$$

$$(c) \quad \frac{d^2y}{dx^2} - 2\frac{dy}{dx} = y.$$

[7 marks]

7. Find a particular solution of the form

$$y = x(a\cos x + b\sin x)$$

for the differential equation

$$\frac{d^2y}{dx^2} + y = \sin x.$$

[7 marks]

8. On October 15, 1997, the Thrust jet car averaged 766 mph ($= 342\,\mathrm{ms}^{-1}$) on its second run over a measured mile. Given that it accelerated uniformly from rest to $342\,\mathrm{ms}^{-1}$ over an approach distance of $13\frac{1}{2}$ miles ($= 21.7\,\mathrm{km}$), calculate this acceleration. A driver is likely to lose consciousness if subjected to an acceleration in excess of about $50\,\mathrm{ms}^{-2}$. Was the Thrust driver in such a danger? [6 marks]

9. A particle of mass 2 kg has velocity vector $(\mathbf{i} + \mathbf{j} + 2\mathbf{k})\,\mathrm{ms}^{-1}$ at time $t = 0$. A constant force $(3\mathbf{i} + 4\mathbf{j} + \mathbf{k})\,\mathrm{N}$ acts on the particle. Find its velocity at time $t = 4\,\mathrm{s}$. [4 marks]

10. At time $t = 0$, a particle of mass 5 kg is at rest at the point with position vector $(2\mathbf{i} + 3\mathbf{j} - \mathbf{k})\,\mathrm{m}$. The particle is subject to a constant force $\mathbf{F}$. At time $t = 10\,\mathrm{s}$, the particle is at the point $(12\mathbf{i} - 2\mathbf{j} + \mathbf{k})\,\mathrm{m}$. Calculate the force $\mathbf{F}$, and the velocity of the particle at time $t = 10\,\mathrm{s}$. [6 marks]

SECTION B

11. My car has a maximum acceleration of $1.5\,\mathrm{ms}^{-2}$ and a maximum deceleration of $2.5\,\mathrm{ms}^{-2}$. If I start from rest, what is the maximum speed that I can achieve after s_1 m? If I then decelerate as hard as possible, find my speed after travelling a further distance s_2 m.

There is a road hump 100 m from my house which I must cross at no more than $2\,\mathrm{ms}^{-1}$ in order not to damage the car. What is the shortest time in which I can get from rest at my house to the hump so as to cross it without damage? [15 marks]

12. Using the substitution $y = xv$, where v is a function of x, transform the differential equation

$$x\frac{dy}{dx} = x^2 + y^2 + y,$$

where $x > 0$, into a differential equation for v.

Solve this equation for v, and hence find y in terms of x, given that $y = 0$ when $x = 1$. You may quote the result

$$\int_0^x \frac{dt}{1+t^2} = \arctan x.$$

[15 marks]

13. An aircraft of mass m lands on the flight deck of a stationary aircraft carrier. It makes its first contact with the deck at time $t = 0$, and at this instant it has speed U. It is brought to rest in a straight line by means of the combined effects of an arrestor cord and the reversed thrust of its engine. At time t let x be the distance of the aircraft down the flight deck from its first point of contact. The retarding force exerted by the arrestor cord is proportional to x and that due to the reversed thrust is proportional to the speed of the aircraft. Show that

$$m\frac{d^2x}{dt^2} = -ax - b\frac{dx}{dt}$$

where a and b are constants, and state the two initial conditions for the function $x(t)$ at $t = 0$.

Given that, in appropriate units, $m = 1, \quad a = b = 2, \quad U = 3$, show that

$$x(t) = 3e^{-t}\sin t.$$

Show that the aircraft comes to rest at time $t = \pi/4$. [15 marks]

14. Solve the following differential equation by separating the variables:

$$\frac{dy}{dx} = x^3y^3.$$

Find the solution y for which $y = 1$ at $x = 2$.

Calculate the derivative of your solution; check to see whether your solution satisfies the differential equation and boundary condition. [15 marks]