PAPER CODE NO.
MATH 013

THE UNIVERSITY *of* LIVERPOOL

JANUARY 2003 EXAMINATIONS

Bachelor of Engineering : Foundation Year
Bachelor of Science : Foundation Year

MATHEMATICAL METHODS

TIME ALLOWED : Three Hours

INSTRUCTIONS TO CANDIDATES

You may attempt all questions. All answers to Section A and the best THREE answers to Section B will be taken into account.
Numerical answers should be given correct to four places of decimals.

SECTION A

1. Determine the radian measure of the angle α of 660^0, expressed as a rational multiple of π. The formula for $\cos(A-B)$ states that

$$\cos(A-B)=\cos(A)\cos(B)+\sin(A)\sin(B).$$

Using this formula or otherwise find the exact value for $\cos(\alpha)$, *without using tables or a calculator*. (Show all your working.)
Hence determine all the angles θ, in the range $[-2\pi, 2\pi]$ satisfying $\cos(\theta)=\cos(\alpha)$. Your answers should be expressed in radian measure.

[6 marks]

2. Sketch the graph of $y=\text{cosec}(x)$ in the range $-2\pi \le x \le 2\pi$. Determine numerically the solutions of $\text{cosec}(x)=4$ in the same range. Express, if possible, your results in radian measure.

[9 marks]

3. Find the domain of *x* for which *both* the functions $\log_2(x)$ and $\log_2(3x+2)$ are defined. Solve the equation

$$\log_2(3x+2)-2\log_2(x)=1.$$

[7 marks]

4. You are given the values of $\log_e(18)=2.890372$ and $\log_e(12)=2.484907$, correct to six decimal places. Obtain the values of the following

$$\log_e(216), \qquad \log_e(6), \qquad \log_e(3),$$

without using tables or a calculator, correct to four decimal places. (Show all your working. HINT: for second part use $6^3=216$.)

[6 marks]

5. Write down the first seven rows of Pascal's triangle. Hence or otherwise find the coefficient of x^3 in the expansion of

$$\left(3x^2-\frac{1}{x}\right)^6.$$

[6 marks]

6. Let $q(x)$ be the quadratic function $q(x) = 6 + 5x - x^2$. Determine the zeros of $q(x)$ and the position of its maximum. Hence sketch the graph of $q(x)$.

[7 marks]

7. Express the rational function $f(x)$ in partial fractions, where

$$f(x) = \frac{4x+11}{(x+5)(x-1)}.$$

[5 marks]

8. Express the complex number

$$z = \frac{2-3\mathrm{i}}{7+2\mathrm{i}}$$

in the form $z = a + b\mathrm{i}$.
Determine numerically the modulus and argument of z. The argument should, preferably, be expressed in radian measure. Hence, or otherwise, find the modulus and argument of z^2.

[9 marks]

SECTION B

9. Given that $t = \tan(\theta/2)$ and $\tan(\theta) = \dfrac{2t}{1-t^2}$, determine two expressions for $\sin(\theta)$ and $\cos(\theta)$ in terms of t.

[7 marks]

Hence, or otherwise, determine all the angles θ, lying in the range $[0, 2\pi]$, which satisfy

$$6\sin(\theta) + 3\cos(\theta) = 2\sqrt{5}\,.$$

[8 marks]

10. (i) On separate diagrams sketch the curves $y = e^{-x}$ for real x, and $y = -\log_e(x)$ for $x > 0$.

[4 marks]

(ii) Solve the following equations:

$$\log_2(x) = 5, \qquad \log_y(27) = 3\,.$$

[4 marks]

(iii) A year long survey of the population of a particular species of fish found in a freshwater lake was carried out. The population $N(t)$ was found to increase, roughly, according to the formula

$$N(t) = \frac{10000}{\alpha + e^{-kt}} \qquad ,$$

where t = time in weeks, k is a (constant) growth rate and α is a constant. Initially it was estimated there were 2000 fish in the lake. Show that the value of $\alpha = 4$. After 10 weeks it was estimated the population had grown by 10%. Determine k (to 4 decimal places) and estimate (to the nearest whole number) the population at the end of the survey (52 weeks).

[7 marks]

11. (i) If α and β are the roots of the equation $3x^2-7x-1=0$, write down the values of a) $\alpha\beta$, b) $\alpha+\beta$, c) $(\alpha-\beta)^2$ and d) $\alpha^2+\beta^2$, *without determining the values of* α *and* β individually.

[8 marks]

(ii) Plot a table of the values of the following cubic polynomial

$$p(x)=3x^3-14x^2+13x+6,$$

for $x=-2,-1,0,1,2,3$ and 4. Sketch the curve of the polynomial, and find all the roots of $p(x)=0$.

[7 marks]

12. (i) A complex number z has modulus one and argument $\pi/4$. Express each of the following complex numbers in the form $a+b\mathrm{i}$:

$$z,\ z^2,\ z^3,\ \frac{1}{z},$$

and plot them on the Argand diagram.

[10 marks]

(ii) Find the real values of a and b such that $(a+b\mathrm{i})^2=\mathrm{i}$. Hence, or otherwise, solve the equation $z^2+2z+1-\mathrm{i}=0$, giving your solutions in the form $z=x+y\mathrm{i}$.

[5 marks]