PAPER CODE NO.
MATH 013

THE UNIVERSITY
of LIVERPOOL

JANUARY 2006 EXAMINATIONS

Bachelor of Engineering : Foundation Year
Bachelor of Science : Foundation Year

MATHEMATICAL METHODS

TIME ALLOWED : **Three Hours**

INSTRUCTIONS TO CANDIDATES

You may attempt all questions. All answers to Section A and the best THREE answers to Section B will be taken into account.
Numerical answers should be given correct to four places of decimals.

SECTION A

1. Determine the radian measure of the angle α of -120^0, expressed as a rational multiple of π.
The formula for $\sin(A-B)$ states that

$$\sin(A-B)=\sin(A)\cos(B)-\cos(A)\sin(B).$$

Using this formula or otherwise find the exact value for $\sin(\alpha)$, *without using tables or a calculator*. (Show all your working.)
Hence determine all the angles θ, in the range [$-360^0, 360^0$] satisfying $\sin(\theta)=\sin(\alpha)$. Your answers can be expressed in degrees or radians.

[6 marks]

2. Sketch the graph of $y=\tan(x)$ in the range $-\pi \le x \le \pi$. Determine numerically the solutions of $\tan(x)=2.5$ and $\tan(x)=-2.5$ in the same range.

[9 marks]

3. Solve the equation

$$\log_e(4x)+\log_e(x^4)=7.$$

[7 marks]

4. You are given the values of $\log_e(100)=4.605170$ and $\log_e(5)=1.609438$, correct to six decimal places. Obtain the values of the following

$$\log_e(500), \qquad \log_e(20), \qquad \log_e(25),$$

without using tables or a calculator, correct to four decimal places. (Show all your working.)

[6 marks]

5. Write down the first six rows of Pascal's triangle. Hence or otherwise find the coefficient of x^4 in the expansion of

$$(1-4x)^5.$$

[6 marks]

6. Let $q(x)$ be the quadratic function $q(x) = 4 - 2x - x^2$. Determine the zeros of $q(x)$ and the position and nature of its turning point. Hence sketch the graph of $q(x)$.

[7 marks]

7. Express the rational function $f(x)$ in partial fractions, where

$$f(x) = \frac{7x - 6}{(x+7)(x-4)}.$$

[5 marks]

8. Express the complex number

$$z = \frac{1 + 3\mathrm{i}}{3 + 4\mathrm{i}}$$

in the form $z = a + b\mathrm{i}$ where a and b are real.
Determine numerically the modulus and argument of z. The argument should, preferably, be expressed in radian measure. Hence, or otherwise, find the modulus and argument of z^2.

[9 marks]

SECTION B

9. Find two values of θ between 0^0 and 180^0 satisfying the equation

$$6\sin^2(\theta) = 4 + \cos(\theta).$$

[7 marks]

Using the identity $\sin(A) + \sin(B) = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$ or otherwise, find the range of values of a for which the equation

$$\sin(x + 225^0) + \sin(x + 135^0) = a\ ,$$

has real solutions. For the case $a = 1/\sqrt{2}$, find all the solutions in the interval $0^0 \le x \le 360^0$.

[8 marks]

10. (i) On separate diagrams sketch the curves $y = \frac{1}{3}e^x$ for real *x*, and $y = \log_e(x) + 2$ for $x > 0$.

[4 marks]

(ii) Solve the following equations:

$$\log_{16}(x) = \frac{1}{2}, \qquad \log_y(121) = 2\ .$$

[4 marks]

(iii) A body of mass $m = 0.02$ kg falls under gravity through a resistive medium which exerts a resistive force proportional to the body's velocity *v*. It turns out that after a time *t* (measured in seconds) *v* is given by the following equation

$$v = \frac{mg}{k} - \frac{A}{k}e^{-kt/m}\ ,$$

where *g* is the acceleration due to gravity (take $g = 9.8\text{ms}^{-2}$) and *k* and *A* are constants. If the body falls initially from rest, show that $A = mg$. As $t \to \infty$ the body's velocity approaches 10ms^{-1}. Calculate the value of *k* and hence show that after 1 second the body is already falling at a speed of approximately 6.25ms^{-1}.

[7 marks]

11. (i) If α and β are the roots of the equation $3x^2+3x+2=0$, find the values of a) $\alpha\beta$, b) $\alpha+\beta$, c) $\alpha^2+\beta^2$ and d) $(\alpha-\beta)^2$, *without determining the values of* α *and* β individually.

[8 marks]

(ii) Plot a table of the values of the following cubic polynomial

$$p(x)=4x^3-8x^2-7x+5,$$

for $x=-2,-1,0,1,2,3$ and 4. Sketch the curve of the polynomial, and find all the roots of $p(x)=0$.

[7 marks]

12. (i) A complex number *z* has modulus one and argument $\pi/4$. Express each of the following complex numbers in the form $a+b\mathrm{i}$ (where *a* and *b* are real):

$$z,\ z^2,\ z^3,\ \frac{1}{z},$$

and plot them on the Argand diagram.

[10 marks]

(ii) Find the modulus and argument of the complex number *w*, where

$$w=\frac{(3+\mathrm{i})(1+\mathrm{i})}{\mathrm{i}}.$$

[5 marks]