PAPER CODE NO.
MATH012

THE UNIVERSITY
of LIVERPOOL

SUMMER 2007 EXAMINATIONS

Bachelor of Engineering : Foundation Year
Bachelor of Science : Foundation Year
Bachelor of Science : Year 1
Bachelor of Science : Year 2

VECTORS AND INTRODUCTION TO STATISTICS

TIME ALLOWED : Three Hours

INSTRUCTIONS TO CANDIDATES

Answer ALL questions in Section A and THREE questions from Section B.
The total of the marks available on Section A is 55.
$\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors along the x, y and z axes respectively.

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SECTION A

1. Let ABC be a triangle. Let D be the mid-point of BC. Given that $\overrightarrow{AB} = \mathbf{u}$ and $\overrightarrow{AC} = \mathbf{v}$, express each of the following in terms of $\mathbf{u}$ and $\mathbf{v}$:

   (a) $\overrightarrow{BC}$

   (b) $\overrightarrow{CB}$

   (c) $\overrightarrow{BD}$

   (d) $\overrightarrow{AD}$.

[**5 marks**]

2. The points P, Q and R have Cartesian coordinates $(1, 2, -1)$, $(3, 4, 0)$ and $(0, 5, 2)$ respectively, where lengths are measured in metres.

   Find:

   (a) the lengths of the sides of triangle PQR, correct to the nearest centimetre

   (b) $\overrightarrow{PQ} \cdot \overrightarrow{PR}$

   (c) the angle $\angle QPR$ in degrees (to the nearest 0.1degree).

   (d) the coordinates of the point S such that $PQRS$ is a parallelogram.

[**13 marks**]

3. Let $\mathbf{u} = 4\mathbf{i} - 2\mathbf{j}$ and $\mathbf{v} = \mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$.

   (a) Compute $2\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - 3\mathbf{v}$.

   (b) Compute $(2\mathbf{u} + \mathbf{v}) \cdot \mathbf{u}$ and $(2\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - 3\mathbf{v})$.

   (c) Find a unit vector parallel to $\mathbf{u} \times \mathbf{v}$.

[**8 marks**]

4. The points A and B have Cartesian coordinates $(1, 2, 1)$ and $(2, 3, 4)$ respectively.

   (a) Compute $\overrightarrow{AB}$

   (b) Find the vector equation of the line $\mathcal{L}$ through A and B.

   (c) Suppose point P has position vector $\mathbf{p} = \mathbf{i} - 2\mathbf{k}$. What is the vector from the point P to a point R on the line $\mathcal{L}$?

   (d) Compute the shortest distance from P to the line $\mathcal{L}$.

[**8 marks**]

5. Let O be the origin of co-ordinates. A particle P moves so that its position vector $\mathbf{r}$ with respect to O at time t is given by

$$\mathbf{r} = a(2t - \sin 2t)\mathbf{i} + a(1 - \cos 2t)\mathbf{j}$$

   where a is a positive constant, t is measured in seconds, distances are measured in metres and both $\sin(2t)$ and $\cos(2t)$ are evaluated by treating $2t$ seconds as $2t$ radians. Find:

   (a) the position of P at time $t = 0$ seconds

   (b) the velocity of P at time t seconds

   (c) the speed of P at $t = \frac{\pi}{4}$ seconds

   (d) the acceleration of P at $t = 0$ seconds.

[**7 marks**]

6. An aircraft sets out from the origin O. The wind velocity relative to the ground is $\mathbf{w} = 50\mathbf{j}$ km/hr where $\mathbf{j}$ is a unit vector pointing North.

   The aircraft travels at a constant velocity $\mathbf{u} = (100\mathbf{i} + 300\mathbf{j})$ km/hr relative to the air. Here $\mathbf{i}$ is a unit vector pointing East.

   (a) Give an expression for the velocity $\mathbf{v}$ of the aircraft relative to the ground.

   (b) Hence write down an expression for the position vector of the aircraft at time t hours.

   (c) Find the time in minutes at which the aircraft has flown 200 km North (relative to the ground).

   (d) Find the position vector of the point P the aircraft reaches after it has flown 200 km North (relative to the ground).

   [**6 marks**]

7. Find the volume of the parallelepiped with edges formed by the vectors $2\mathbf{i}+3\mathbf{j}-\mathbf{k}$, $\mathbf{i}-\mathbf{j}+4\mathbf{k}$, and $\mathbf{i}+3\mathbf{j}+\mathbf{k}$.

   [**4 marks**]

8. What does the conditional probability $P(X|Y)$ of events X and Y mean?

   Two machines A and B make mobile phones. In a given batch at the factory, 15% of the phones are made by A and 85% by B. Also, 60% of the phones made by A are acceptable, and 20% of the phones made by B are acceptable. What is the probability that a given phone chosen at random from the whole batch is acceptable?

   [**4 marks**]

SECTION B

9. The four distinct points A, B, C and D are non-collinear and such that $\overrightarrow{AB} = \mathbf{u}$, $\overrightarrow{BC} = \mathbf{v}$ and $\overrightarrow{DA} = \mathbf{w}$. Suppose that

$$\mathbf{u} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}, \quad , \quad \mathbf{v} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k} \quad \text{and} \quad \mathbf{w} = 2\mathbf{i} + 5\mathbf{j} + 5\mathbf{k}.$$

(a) Using the vectors $\mathbf{u}$ and $\mathbf{v}$, show that a unit vector normal to the plane containing the points A, B and C is given by

$$\hat{\mathbf{n}} = \frac{5\mathbf{i} + \mathbf{j} - 3\mathbf{k}}{\sqrt{35}}.$$

(b) Verify by explicitly calculating the scalar products that

$$\hat{\mathbf{n}} \cdot \mathbf{u} = 0 \quad \text{and} \quad \hat{\mathbf{n}} \cdot \mathbf{v} = 0.$$

(c) Show that A, B, C and D lie in the same plane.

(d) Suppose that A is the point $(2, 3, 1)$. What is the Cartesian equation of the plane through A, B, C and D?

(e) Find the line of intersection of the plane through A, B, C and D with the plane $x + y + z = 6$.

[**15 marks**]

10. Suppose that the line $\mathcal{L}_1$ has vector equation

$$\mathbf{r} = \mathbf{i} - \mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + 5\mathbf{k})$$

and that the line $\mathcal{L}_2$ has vector equation

$$\mathbf{r} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k} + \mu(-2\mathbf{i} - 2\mathbf{j} + 12\mathbf{k}).$$

(a) Write down the coordinates of *any* two points on the line $\mathcal{L}_1$.

(b) Determine two unit vectors $\hat{\mathbf{u}}_1$ and $\hat{\mathbf{u}}_2$ which are respectively parallel to the lines $\mathcal{L}_1$ and $\mathcal{L}_2$.

(c) Compute the angle between the lines (in radians, to 3 sig. fig.).

(d) Show that the lines intersect and find the coordinates of the point of intersection.

[15 marks]

11. The planes Π_1, Π_2 and Π_3 have equations

$$x + y - 2z = 12, \qquad 3x - y - z = 5, \qquad \text{and} \qquad 2x + 4y + z = 6,$$

respectively.

(a) Find a normal to each plane.

(b) Find the angle in degrees (to the nearest degree) between the normals to the planes Π_1 and Π_2.

(c) Find the point of intersection of Π_1, Π_2 and Π_3.

[15 marks]

12. Define the mean, mode and median of a set of values.

    The maximum temperature on the first day of the month for 12 consecutive months is measured (in degrees Centigrade) as

$$3, 2, 3, 8, 15, 18, 16, 20, 14, 11, 9, 3$$

    (a) Draw a bar chart to show the number of days with maximum temperature in the ranges 0-5, 6-10, 11-15 and 16-20 degrees.
    (b) What is the frequency and relative frequency of a result of 3 degrees?
    (c) What is the mean maximum temperature in degrees?
    (d) What are the mode and median?
    (e) What is the standard deviation of the maximum temperature (to the nearest 0.01 degrees)?

[15 marks]