MATH012 May 2006 Exam Solutions

1.

(a) $\overrightarrow{CD} = -\mathbf{u}$.
(b) $\overrightarrow{BD} = -\mathbf{u} - \mathbf{v}$.
(c) $\overrightarrow{BP} = \mathbf{v} - \frac{1}{2}\mathbf{u}$.

2.

(a)

$$\begin{aligned}\overrightarrow{QP} =& 4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \Rightarrow |\overrightarrow{PQ}| = \sqrt{16+4+4} = \sqrt{24}\\ \overrightarrow{QR} =& 4\mathbf{i} - \mathbf{j} - 2\mathbf{k} \Rightarrow |\overrightarrow{QR}| = \sqrt{16+1+4} = \sqrt{21}\\ \overrightarrow{RP} =& 3\mathbf{j} + 4\mathbf{k} \Rightarrow |\overrightarrow{RP}| = \sqrt{9+16} = \sqrt{25} = 5.\end{aligned}$$

(b)

$$\overrightarrow{QP}.\overrightarrow{RP} = 2.3 + 2.4 = 6 + 8 = 14.$$

(c)

$$\cos R\hat{P}Q = \frac{\overrightarrow{QP}.\overrightarrow{RP}}{|\overrightarrow{QP}||\overrightarrow{RP}|} = \frac{14}{5\sqrt{24}} \Rightarrow Q\hat{P}R = 0.96 \quad (55.1\,\text{deg})$$

(d)

$$\mathbf{s} = \mathbf{p} + \overrightarrow{PS} = \mathbf{p} + \overrightarrow{QR} = 2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k} + (4\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = 6\mathbf{i} + 3\mathbf{j} + \mathbf{k}.$$

So S is $(6, 3, 1)$.

3(a).

$$\begin{aligned}3\mathbf{u}+\mathbf{v} =&3(2\mathbf{i}-3\mathbf{j}+\mathbf{k})+(\mathbf{i}+4\mathbf{j}-2\mathbf{k}) = 7\mathbf{i}-5\mathbf{j}+\mathbf{k}\\ \mathbf{u}-2\mathbf{v} =&2\mathbf{i}-3\mathbf{j}+\mathbf{k}-2(\mathbf{i}+4\mathbf{j}-2\mathbf{k}) = -11\mathbf{j}+5\mathbf{k}.\end{aligned}$$

(b)

$$\begin{aligned}(3\mathbf{u}+\mathbf{v}).\mathbf{v} =&(7\mathbf{i}-5\mathbf{j}+\mathbf{k}).(\mathbf{i}+4\mathbf{j}-2\mathbf{k}) = 7.1-5.4+1.(-2) = -15\\ (3\mathbf{u}+\mathbf{v}).(\mathbf{u}-2\mathbf{v}) =&(7\mathbf{i}-5\mathbf{j}+\mathbf{k}).(-11\mathbf{j}+5\mathbf{k}) = -5.(-11)+1.5 = 60.\end{aligned}$$

(c)

$$\begin{aligned}\mathbf{u}\times\mathbf{v} =&\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & -3 & 1\\ 1 & 4 & -2\end{vmatrix}\\ =&2\mathbf{i}+5\mathbf{j}+11\mathbf{k}.\\ |\mathbf{u}\times\mathbf{v}| =&\sqrt{4+25+121} = \sqrt{150} = 5\sqrt{6}.\end{aligned}$$

So a unit vector in the direction of $\mathbf{u}\times\mathbf{v}$ is given by

$$\frac{1}{|\mathbf{u}\times\mathbf{v}|}\mathbf{u}\times\mathbf{v} = \frac{1}{5\sqrt{6}}(2\mathbf{i}+5\mathbf{j}+11\mathbf{k}).$$

4.(a) $\overrightarrow{AB} = 2\mathbf{i}+\mathbf{j}-2\mathbf{k}$.

(b)

$$\mathbf{r} = \mathbf{a}+\lambda\overrightarrow{AB} = \mathbf{i}+2\mathbf{j}+3\mathbf{k}+\lambda(2\mathbf{i}+\mathbf{j}-2\mathbf{k}) = (1+2\lambda)\mathbf{i}+(2+\lambda)\mathbf{j}+(3-2\lambda)\mathbf{k}.$$

(c)

$$\overrightarrow{PR} = \mathbf{r}-\mathbf{p} = (2+2\lambda)\mathbf{i}+(2+\lambda)\mathbf{j}-(3+2\lambda)\mathbf{k}.$$

(d) Let $d = |\overrightarrow{RP}|$. Then

$$\begin{aligned}d^2 =&(2+2\lambda)^2+(2+\lambda)^2+(3+2\lambda)^2 = 9\lambda^2+24\lambda+17\\ \Rightarrow \frac{d(d^2)}{d\lambda} =&18\lambda+24 = 0 \quad \text{when} \quad \lambda = -\frac{4}{3}.\\ \Rightarrow d^2 =&9.\frac{16}{9}+24.\left(-\frac{4}{3}\right)+17 = 1 \Rightarrow d = 1.\end{aligned}$$

5.(a) $\mathbf{r}(0) = -\mathbf{i}$ so at $t = 0$ P is at $(-1, 0, 0)$.

(b)

$$\dot{\mathbf{r}} = 2\mathbf{i}+[\sin(2t)+2t\cos(2t)]\mathbf{j}+4t\mathbf{k}.$$

(c)

$$\begin{aligned}\dot{\mathbf{r}}(\frac{\pi}{4}) =&2\mathbf{i}+\mathbf{j}+\pi\mathbf{k} \Rightarrow |\dot{\mathbf{r}}(\frac{\pi}{4})|^2 = 4+1+\pi^2 = 5+\pi^2\\ \Rightarrow \text{Speed} =&|\dot{\mathbf{r}}(\frac{\pi}{4})| = \sqrt{5+\pi^2} \approx 3.86\text{ms}^{-1}.\end{aligned}$$

(d)

$$\ddot{\mathbf{r}} = [4\cos(2t) - 4t\sin(2t)]\mathbf{j} + 4\mathbf{k} \Rightarrow \ddot{\mathbf{r}}(0) = 4(\mathbf{j} + \mathbf{k}).$$

6.

(a)

$$\mathbf{v} = \mathbf{u} + \mathbf{w} = -100\mathbf{i} + 400\mathbf{j}.$$

(b)

$$\mathbf{r} = (-100\mathbf{i} + 400\mathbf{j})t + \mathbf{c}$$

where $\mathbf{c}$ is a constant vector. But $\mathbf{r}(0) = \mathbf{0}$, so $\mathbf{c} = \mathbf{0}$. Hence

$$\mathbf{r} = (-100\mathbf{i} + 400\mathbf{j})t.$$

(c) Time taken to fly 160km North $= \frac{160}{400} = \frac{2}{5}$hours= 24mins.

(d) When $t = \frac{2}{5}$, $\mathbf{r} = (-100\mathbf{i} + 400\mathbf{j})\frac{2}{5} = -40\mathbf{i} + 160\mathbf{j}$. So $\mathbf{p} = -40\mathbf{i} + 160\mathbf{j}$.

7.

$$\begin{vmatrix} 3 & 1 & -2 \\ 1 & -2 & 3 \\ 2 & 3 & -1 \end{vmatrix} = 3(2-9) - (-1-6) - 2(3+4) = -28.$$

So volume is 28 units.

8. $P(X|Y)$ means the probability that X occurs given that Y occurs.
Let X be the event that a component is acceptable.
Let A be the event that the component was made by machine A.
Let B be the event that the component was made by machine B.
Then $P(A) = 0.1$, $P(B) = 0.9$, $P(X|A) = 0.8$, $P(X|B) = 0.6$. We have

$$\begin{aligned} P(X) =& P(X|A)P(A) + P(X|B)P(B) \\ =& 0.8 \times 0.1 + 0.6 \times 0.9 = 0.62. \end{aligned}$$

9.

(a) Normal given by

$$\begin{aligned}\mathbf{n} =& \mathbf{u} \times \mathbf{v} \\ =& \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 1 & -3 & -1 \end{vmatrix} = 4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} \\ |\mathbf{n}| =& 2\sqrt{2^2 + 1^2 + 1^2} = 2\sqrt{6}\end{aligned}$$

$$\Rightarrow \hat{\mathbf{n}} = \frac{1}{\sqrt{6}}(2\mathbf{i} + \mathbf{j} - \mathbf{k}).$$

(b)

$$\hat{\mathbf{n}}.\mathbf{u} = \frac{1}{\sqrt{6}}(2.1 + 1.(-1) + (-1).1) = 0; \quad \hat{\mathbf{n}}.\mathbf{v} = \frac{1}{\sqrt{6}}(2.1 + 1.(-3) + (-1).(-1)) = 0.$$

(c) A, B, C and D will all lie in the same plane if $\overrightarrow{DA}.\hat{\mathbf{n}} = \mathbf{w}.\hat{\mathbf{n}} = 0$. We have

$$\mathbf{w}.\hat{\mathbf{n}} = \frac{1}{\sqrt{6}}(2.3 + 1.(-2) + (-1).4) = 0.$$

So the points do lie in the same plane.

(d) Since the plane passes through $(2, 3, 1)$, the equation is

$$2x + y - z = 2.2 + 3 - 1 = 6.$$

(e) Intersection where

$$2(1 + \lambda) + (-2 - 3\lambda) - (4 + 4\lambda) = 6 \Rightarrow -5\lambda = 10 \Rightarrow \lambda = -2.$$

So intersection is $(-1, 4, -4)$.

10.(a) Two points on $\mathcal{L}_1$ are $(1, 1, 2)$ $(\lambda = 0)$ and $(2, 0, 0)$ $(\lambda = 1)$.

(b) A vector along $\mathcal{L}_1$ is $\mathbf{u}_1 = \mathbf{i} - \mathbf{j} - 2\mathbf{k}$. Have $|\mathbf{u}_1| = \sqrt{1+1+4} = \sqrt{6}$ so

$$\hat{\mathbf{u}}_1 = \frac{1}{|\mathbf{u}_1|}\mathbf{u}_1 = \frac{1}{\sqrt{6}}(\mathbf{i} - \mathbf{j} - 2\mathbf{k}).$$

A vector along $\mathcal{L}_2$ is $\mathbf{u}_2 = \mathbf{i} - \mathbf{k}$. Have $|\mathbf{u}_2| = \sqrt{1+1} = \sqrt{2}$ so

$$\hat{\mathbf{u}}_2 = \frac{1}{|\mathbf{u}_2|}\mathbf{u}_2 = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{k}).$$

(c) The angle θ between the lines is the angle between $\hat{\mathbf{u}}_1$ and $\hat{\mathbf{u}}_2$. So

$$\cos\theta = \frac{\hat{\mathbf{u}}_1.\hat{\mathbf{u}}_2}{|\hat{\mathbf{u}}_1||\hat{\mathbf{u}}_2|} = \frac{3}{\sqrt{6}.\sqrt{2}} = \frac{3}{\sqrt{12}} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{6}.$$

The lines intersect if there is a solution to

$$\begin{aligned} 1 + \lambda =& 2 + \mu, \\ 1 - \lambda =& -1, \\ 2 - 2\lambda =& -1 - \mu. \end{aligned}$$

These equations have the solution $\lambda = 2$, $\mu = 1$ so the point of intersection is $(3, -1, -2)$.

11.(a) The normals are given by

$$\mathbf{n}_1 = \mathbf{i} - \mathbf{j} - 4\mathbf{k}, \quad \mathbf{n}_2 = 4\mathbf{i} - \mathbf{j} - \mathbf{k}, \quad \mathbf{n}_3 = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}.$$

(b) The angle θ between $\mathbf{n}_1$ and $\mathbf{n}_2$ is given by

$$\cos\theta = \frac{\mathbf{n}_1.\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{9}{\sqrt{18}\sqrt{18}} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}.$$

(c) The planes intersect where

$$\begin{aligned}
x - y - 4z =& 13 \\
4x - y - z =& 10 \\
3x + 2y + z =& 5 \\
\Rightarrow \begin{bmatrix} 1 & -1 & -4 & 13 \\ 4 & -1 & -1 & 10 \\ 3 & 2 & 1 & 5 \end{bmatrix} & \\
\Rightarrow \begin{bmatrix} 1 & -1 & -4 & 13 \\ 0 & 3 & 15 & -42 \\ 0 & 5 & 13 & -34 \end{bmatrix} & \\
\Rightarrow \begin{bmatrix} 1 & -4 & 1 & 12 \\ 0 & 1 & 5 & -14 \\ 0 & 0 & -12 & 36 \end{bmatrix} & \\
\Rightarrow z = -3 \quad y + 5z = -14 \Rightarrow y = 1 & \\
x - y - 4z =& 13 \Rightarrow x = 2.
\end{aligned}$$

12. The mean is the sum of the values divided by the number of values.

The mode is the value that occurs most often.

The median is found by forming a list of the values in ascending order and then selecting the value that lies halfway along the list. (For an *even* set of values, the median is the mean of the two values on either side of the halfway point.]

(a) Rainfalls in order:

$$0, 1, 1, 2, 2, 3, 3, 5, 5, 5, 7, 7, 10, 11, 13$$

(b) Frequency of $7 = 2$. Relative frequency of $7 = \frac{2}{15}$.

(c)

$$\text{Mean} = \bar{x} = \frac{1+1+2+2+3+3+5+5+5+7+7+10+11+13}{15} = \frac{75}{15} = 5$$

(d) Mode is 5; Median is 5.

(e) Standard deviation given by

$$\sigma^2 = \frac{\sum(x_i - \bar{x})^2}{n} = \frac{191}{15} = 12.73 \Rightarrow \sigma = 3.57.$$