PAPER CODE NO.
**MATH012**

THE UNIVERSITY
*of* LIVERPOOL

# SUMMER 2005 EXAMINATIONS

Bachelor of Engineering : Foundation Year
Bachelor of Science : Foundation Year
Bachelor of Science : Year 1
Bachelor of Science : Year 2

## VECTORS AND INTRODUCTION TO STATISTICS

TIME ALLOWED : Three Hours

---

INSTRUCTIONS TO CANDIDATES

Answer ALL questions in Section A and THREE questions from Section B.
The total of the marks available on Section A is 55.
$\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors along the $x$, $y$ and $z$ axes respectively.

---

THE UNIVERSITY
*of* LIVERPOOL

## SECTION A

1. Let $ABCD$ be a parallelogram. Given that $\overrightarrow{AB} = \mathbf{u}$ and $\overrightarrow{BC} = \mathbf{v}$, express each of the following in terms of $\mathbf{u}$ and $\mathbf{v}$:

   (a) $\overrightarrow{DA}$

   (b) $\overrightarrow{BD}$

   (c) $\overrightarrow{BP}$, where $P$ is the mid-point of $\overrightarrow{AD}$.

**[5 marks]**

2. The points $P$, $Q$ and $R$ have Cartesian coordinates $(4, 3, 1)$, $(2, 1, -2)$ and $(2, -1, 4)$ respectively, where lengths are measured in metres.

   Find:

   (a) the lengths of the sides of triangle $PQR$, correct to the nearest centimetre

   (b) $\overrightarrow{QP} \cdot \overrightarrow{RP}$

   (c) the angle $\angle QPR$ in degrees

   (d) the coordinates of the point $S$ such that $PQRS$ is a parallelogram.

**[13 marks]**

3. Let $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$.

   (a) Compute $2\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - 2\mathbf{v}$.

   (b) Compute $(2\mathbf{u} + \mathbf{v}) \cdot \mathbf{v}$ and $(2\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - 2\mathbf{v})$.

   (c) Find a unit vector parallel to $\mathbf{u} \times \mathbf{v}$.

**[8 marks]**

4. The points $A$ and $B$ have Cartesian coordinates $(2,2,2)$ and $(4,4,1)$ respectively.

   (a) Compute $\overrightarrow{AB}$

   (b) Find the vector equation of the line $\mathcal{L}$ through $A$ and $B$

   (c) Suppose point $P$ has position vector $\mathbf{p} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$. What is the vector from the point $P$ to a point $R$ on the line $\mathcal{L}$ ?

   (d) Compute the shortest distance from $P$ to the line $\mathcal{L}$.

[**8 marks**]

5. Let $O$ be the origin of co-ordinates. A particle $P$ moves so that its position vector $\mathbf{r}$ with respect to $O$ at time $t$ is given by

$$\mathbf{r} = te^{-t}\mathbf{i} + t^2\mathbf{j} + (t-4)\mathbf{k}$$

   where $t$ is measured in seconds and distances are measured in metres. Find:

   (a) the position of $P$ at time $t = 0$ seconds

   (b) the velocity of $P$ at time $t$ seconds

   (c) the speed of $P$ at $t = 2$ seconds, to the nearest cm/sec

   (d) the acceleration of $P$ at $t = 0$ seconds.

[**6 marks**]

6. A ferry boat sets out from an origin $O$ on the bank to cross a river flowing with constant velocity $\mathbf{w} = 2\mathbf{i}$ km/hr, where $\mathbf{i}$ is a unit vector parallel to the river. The ferry boat travels at constant velocity $\mathbf{u} = \mathbf{i} + 3\mathbf{j}$ km/hr relative to the river. Here $\mathbf{j}$ is a unit vector orthogonal to the river flow.

   (a) Give an expression for the velocity $\mathbf{v}$ of the ferry relative to the land.

   (b) Hence write down an expression for the position vector of the ferry at time $t$ hours.

   (c) If the river is 2km wide, find the time in minutes at which the ferry reaches the opposite side.

   (d) Find the position vector of the point $P$ at which the ferry reaches the opposite side.

[**6 marks**]

7. Find the volume of the parallelepiped with edges formed by the vectors $2\mathbf{i}+\mathbf{j}-\mathbf{k}$, $\mathbf{i}-2\mathbf{j}+\mathbf{k}$, and $\mathbf{i}-\mathbf{j}+2\mathbf{k}$.

[**4 marks**]

8. What does the conditional probability $P(X|Y)$ of events $X$ and $Y$ mean?

   Two machines $A$ and $B$ make components for cars. In a given batch at the car factory, 80% of components are made by $A$ and 20% by $B$. Also, 70% of components made by $A$ are acceptable, and 90% of components made by $B$ are acceptable. What is the probability that a given component chosen at random from the whole batch is acceptable?

[**5 marks**]

## SECTION B

9. The four distinct points $A$, $B$, $C$ and $D$ are non-collinear and such that $\overrightarrow{AB} = \mathbf{u}$, $\overrightarrow{BC} = \mathbf{v}$ and $\overrightarrow{DA} = \mathbf{w}$. Suppose that

$$\mathbf{u} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}, \quad , \quad \mathbf{v} = 3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} \quad \text{and} \quad \mathbf{w} = 2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}.$$

(a) Using the vectors $\mathbf{u}$ and $\mathbf{v}$, show that a unit vector normal to the plane containing the points $A$, $B$ and $C$ is given by

$$\hat{\mathbf{n}} = \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3}.$$

(b) Show by explicitly calculating the scalar products that

$$\hat{\mathbf{n}} \cdot \mathbf{u} = 0 \quad \text{and} \quad \hat{\mathbf{n}} \cdot \mathbf{v} = 0.$$

(c) Show that $A$, $B$, $C$ and $D$ lie in the same plane.

(d) Suppose that $A$ is the point $(1, 1, 2)$. What is the Cartesian equation of the plane through $A$, $B$, $C$ and $D$?

(e) Find where the straight line

$$\mathbf{r} = -\mathbf{i} + \mathbf{j} + 4\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} - 2\mathbf{k})$$

intersects the plane through $A$, $B$, $C$ and $D$.

**[15 marks]**

10. Suppose that the line $\mathcal{L}_1$ has vector equation

$$\mathbf{r} = \mathbf{i} + 5\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

and that the line $\mathcal{L}_2$ has vector equation

$$\mathbf{r} = 3\mathbf{i} + 3\mathbf{j} - \mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} - \mathbf{k}).$$

(a) Write down the coordinates of *any* two points on the line $\mathcal{L}_1$.

(b) Determine two unit vectors $\hat{\mathbf{u}}_1$ and $\hat{\mathbf{u}}_2$ which are respectively parallel to the lines $\mathcal{L}_1$ and $\mathcal{L}_2$.

(c) Show that the angle between the lines is 60 degrees.

(d) Show that the lines intersect and find the coordinates of the point of intersection.

[**15 marks**]

11. The planes $\Pi_1$, $\Pi_2$ and $\Pi_3$ have equations

$$x - 4y + z = 12, \qquad 2x - 2y - z = 9, \qquad \text{and} \qquad x + y + 2z = 3,$$

respectively.

(a) Find a normal to each plane.

(b) Find the angle in degrees between the normals to the planes $\Pi_1$ and $\Pi_2$.

(c) Find the point of intersection of $\Pi_1$, $\Pi_2$ and $\Pi_3$.

[**15 marks**]

12. Define the mean, mode and median of a set of values.

    12 students sit an exam with marks given by

$$6, 3, 2, 5, 8, 10, 12, 8, 3, 1, 14, 8.$$

   (a) Draw a bar chart to show the number of students with marks in the ranges 0-5, 6-10 and 11-15.
   (b) What is the frequency and relative frequency of a result of 3?
   (c) What is the mean mark?
   (d) What are the mode and median?
   (e) What is the standard deviation of the marks?

[**15 marks**]