PAPER CODE NO.
**MATH012**

THE UNIVERSITY
*of* LIVERPOOL

# SUMMER 2003 EXAMINATIONS

Bachelor of Engineering : Foundation Year
Bachelor of Science : Foundation Year
Bachelor of Science : Year 1
Bachelor of Science : Year 2

## VECTORS AND INTRODUCTION TO STATISTICS

TIME ALLOWED : Three Hours

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INSTRUCTIONS TO CANDIDATES

Answer ALL questions in Section A and THREE questions from Section B.
The total of the marks available on Section A is 55.

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## SECTION A

1. Let $ABCD$ be a parallelogram. Given that $\overrightarrow{AB} = \mathbf{u}$ and $\overrightarrow{BC} = \mathbf{v}$, express each of the following in terms of $\mathbf{u}$ and $\mathbf{v}$:

   (a) $\overrightarrow{CD}$

   (b) $\overrightarrow{BD}$

   (c) $\overrightarrow{BP}$, where $P$ is the mid-point of $\overrightarrow{CD}$.

[**4 marks**]

2. The points $P$, $Q$ and $R$ have Cartesian coordinates (2,3,1), (3,1,-1) and (4,2,3) respectively, where lengths are measured in metres.

   Find:

   (a) the lengths of the sides of triangle $PQR$, correct to the nearest centimetre

   (b) $\overrightarrow{PQ} \cdot \overrightarrow{PR}$

   (c) the angles of the triangle $PQR$ in degrees

   (d) the coordinates of the point $S$ such that $PQSR$ is a parallelogram.

[**12 marks**]

3. Let $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$ where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are mutually orthogonal unit vectors.

   (a) Find $\mathbf{u} + 2\mathbf{v}$ and $\mathbf{u} - 2\mathbf{v}$.

   (b) Find $(\mathbf{u} + 2\mathbf{v}) \cdot \mathbf{u}$ and $(\mathbf{u} + 2\mathbf{v}) \cdot (\mathbf{u} - 2\mathbf{v})$.

   (c) A unit vector parallel to $\mathbf{u} \times \mathbf{v}$.

   (d) $(\mathbf{u} + \mathbf{v}) \times (\mathbf{u} - \mathbf{v})$.

[**9 marks**]

4. The points $A$ and $B$ have Cartesian coordinates (1,2,3) and (2,-1,-1) respectively. Find:

   (a) $\overrightarrow{AB}$

   (b) the vector equation of the line through $A$ and $B$

   (c) the coordinates of the point at 2/3 of the distance along the line from $A$ to $B$.

[**5 marks**]

5. Let $O$ be a fixed origin and let $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ be constant, mutually orthogonal unit vectors. A particle $P$ moves so that its position vector $\mathbf{r}$ with respect to $O$ at time $t$ is given by

$$\mathbf{r} = t^2\mathbf{i} + (t-1)\mathbf{j} + te^t\mathbf{k}$$

   where $t$ is measured in seconds and distances are measured in metres. Find:

   (a) the position of $P$ at time $t = 0$ seconds

   (b) the velocity of $P$ at time $t$ seconds

   (c) the speed of $P$ at $t = 2$ seconds, to the nearest cm/sec

   (d) the acceleration of $P$ at $t = 0$.

[**7 marks**]

6. A ferry boat sets out from an origin $O$ on the bank to cross a river flowing with constant velocity $\mathbf{w} = 3\mathbf{i}$ km/hr, where $\mathbf{i}$ is a unit vector parallel to the river. The ferry boat travels at constant velocity $\mathbf{u} = \mathbf{i} + 2\mathbf{j}$ km/hr relative to the river. Here $\mathbf{j}$ is a unit vector orthogonal to the river flow.

   (a) Give an expression for the velocity $\mathbf{v}$ of the ferry relative to the land.

   (b) Hence write down an expression for the position vector of the ferry at time $t$ hours.

   (c) If the river is 0.5km wide, find the time in minutes at which the ferry reaches the opposite side.

   (d) Find the position vector of the point $P$ at which the ferry reaches the opposite side, giving distances to the nearest metre.

[**7 marks**]

7. Vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are each non-zero, none is parallel to any other, but are such that

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 0\,.$$

   Use the geometrical interpretation of the triple scalar product to deduce what you can about these three vectors, clearly stating your reasons.

[**4 marks**]

8. What does the conditional probability $P(X|Y)$ of events $X$ and $Y$ mean?

   The probability of a sunny day in Liverpool is 0.1. The probability of a sunny day in Athens is 0.9. Businessman $X$ spends each day of the year in either Liverpool or Athens. If $X$ is twice as likely to be in Liverpool as in Athens on any given day, what is the probability that $X$ will have sun on a given day?

[**7 marks**]

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## SECTION B

9. The four distinct points $A$, $B$, $C$ and $D$ are non-collinear and such that $\overrightarrow{AB} = \mathbf{u}$, $\overrightarrow{BC} = \mathbf{v}$ and $\overrightarrow{CD} = \mathbf{w}$.

   (a) Find an expression for $\overrightarrow{DA}$ in terms of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

   (b) What condition should be satisfied by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in order that $ABCD$ should be a parallelogram with $AB$ and $DC$ as opposite sides?

   (c) Suppose that, in terms of mutually orthogonal unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$,

   $$\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}, \quad , \quad \mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k} \quad \text{and} \quad \mathbf{w} = -2\mathbf{i} + 3\mathbf{j} + \mathbf{k}.$$

   (i) Show that $ABCD$ is not a parallelogram.

   (ii) Using the vectors $\mathbf{u}$ and $\mathbf{v}$, show that a unit vector normal to the plane containing the points $A$, $B$ and $C$ is given by

   $$\mathbf{n} = \frac{-3\mathbf{i} - 2\mathbf{j}}{\sqrt{13}}.$$

   (iii) Show by explicitly calculating the scalar products that

   $$\mathbf{n} \cdot \mathbf{u} = 0 \quad \text{and} \quad \mathbf{n} \cdot \mathbf{v} = 0.$$

   (iv) Show that $A$, $B$, $C$ and $D$ lie in the same plane.

**[15 marks]**

10. Suppose that the line $\mathcal{L}_1$ has vector equation

$$\mathbf{r} = \mathbf{i} + \mathbf{j} + \lambda(-\mathbf{i} + \mathbf{j})$$

and that the line $\mathcal{L}_2$ has vector equation

$$\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + \mu(-2\mathbf{i} + \mathbf{j} - \mathbf{k})$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors parallel to the coordinate axes $Ox$, $Oy$ and $Oz$.

(a) Write down the coordinates of *any* two points on the line $\mathcal{L}_1$.

(b) Determine two unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ which are respectively parallel to the lines $\mathcal{L}_1$ and $\mathcal{L}_2$.

(c) Show that the angle between the lines is 30 degrees.

(d) Show that the lines intersect and find the coordinates of the point of intersection.

**[15 marks]**

11. The vectors

$$\mathbf{n}_1 = \mathbf{i} - 2\mathbf{k} \quad \text{and} \quad \mathbf{n}_2 = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$$

are respectively normal to the planes $\Pi_1$ and $\Pi_2$. Here $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the usual unit vectors parallel to the $x$, $y$ and $z$ axes respectively. The points $P$ and $Q$ with position vectors

$$\mathbf{p} = \mathbf{i} - \mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{q} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$$

lie on $\Pi_1$ and $\Pi_2$ respectively.

(a) Show that the scalar equations of $\Pi_1$ and $\Pi_2$ can be written as

$$x - 2z = -1 \quad \text{and} \quad 2x + y - 3z = -3.$$

(b) Find the acute angle in degrees between the normals to the planes $\Pi_1$ and $\Pi_2$

(c) Show that the vector equation of the line $\mathcal{L}$ of intersection of the planes $\Pi_1$ and $\Pi_2$ can be written in the form

$$\mathbf{r} = -\mathbf{i} - \mathbf{j} + \lambda(2\mathbf{i} - \mathbf{j} + \mathbf{k}) \ .$$

(d) Verify that the direction of this line is perpendicular to $\mathbf{n}_1$ and $\mathbf{n}_2$.

**[15 marks]**

12. Define the mean, mode and median of a set of values.

12 students sit an exam with marks given by

$$3, 2, 5, 10, 4, 1, 15, 10, 3, 10, 12, 9.$$

(a) Draw a bar chart to show the number of students with marks in the ranges 0-5, 6-10 and 11-15.

(b) What is the frequency and relative frequency of a result of 3?

(c) What is the mean mark?

(d) What is the mode and median?

(e) What is the standard deviation of the marks?

**[15 marks]**