PAPER CODE NO.
MATH012

THE UNIVERSITY
of LIVERPOOL

SUMMER 2002 EXAMINATIONS

Bachelor of Engineering : Foundation Year
Bachelor of Science : Foundation Year
Bachelor of Science : Year 1
Bachelor of Science : Year 2

VECTORS AND KINEMATICS

TIME ALLOWED : Three Hours

INSTRUCTIONS TO CANDIDATES

Answer ALL questions in Section A and THREE questions from Section B.
The total of the marks available on Section A is 55.

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SECTION A

1. In parallelogram $ABCD$, the sides AB and AD are given by the vectors $\mathbf{u}$ and $\mathbf{v}$ respectively. The points L and M are the midpoints of the sides AB and BC respectively. Find expressions for the following in terms of $\mathbf{u}$ and $\mathbf{v}$:

   (a) $\overrightarrow{BM}$

   (b) $\overrightarrow{AL}$

   (c) $\overrightarrow{LM}$.

[4 marks]

2. The points P, Q and R have Cartesian coordinates $(1, 3, 0)$, $(2, 1, -2)$ and $(3, 2, 2)$ respectively where lengths are measured in centimetres.

   Find:

   (a) $\overrightarrow{PQ}$

   (b) $\overrightarrow{QR}$

   (c) the coordinates of the point S such that $PQRS$ is a parallelogram

   (d) the total length of the sides of parallelogram $PQRS$ in centimetres to the nearest millimetre

   (e) $\overrightarrow{QP} \cdot \overrightarrow{QR}$

   (f) all angles of the parallelogram $PQRS$.

[13 marks]

3. Let $\mathbf{u} = -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are mutually orthogonal unit vectors.

   (a) Find $\mathbf{u} + 2\mathbf{v}$ and $\mathbf{u} - 2\mathbf{v}$.

   (b) Find $(2\mathbf{v} - \mathbf{u}) \cdot \mathbf{v}$ and $(\mathbf{u} + 2\mathbf{v}) \cdot \mathbf{u}$.

   (c) Show, by explicitly calculating the vector product, that

$$(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u} + 2\mathbf{v}) = 0 \ .$$

[8 marks]

4. The straight line $\mathcal{L}$ has vector equation

$$\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \lambda(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors parallel to the x, y and z axes respectively and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

(a) Find a unit vector parallel to $\mathcal{L}$.

(b) Suppose point P has position vector $\mathbf{p} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. What is the vector from a point R on the line $\mathcal{L}$ to the point P ?

(c) Find the coordinates of the point on the line $\mathcal{L}$ closest to the point P.

[**8 marks**]

5. Let O be a fixed origin and let $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ be constant, mutually orthogonal unit vectors. A particle P moves so that its position vector $\mathbf{r}$ with respect to O at time t is given by

$$\mathbf{r} = (3 - 2t)\mathbf{i} + 2t^2\mathbf{j} - \cos(\pi t/3)\mathbf{k}$$

where t is measured in seconds, distances are measured in metres and $\cos(\pi t/3)$ is evaluated by treating $\pi t/3$ seconds as $\pi t/3$ radians. Find:

(a) the position of P at time $t = 3$ seconds

(b) the velocity of P at time t seconds

(c) the speed of P at $t = 3$ seconds, to the nearest cm/sec

(d) the acceleration of P at $t = 0$.

[**7 marks**]

6. An aircraft sets out from the origin O. The wind velocity relative to the ground is $\mathbf{w} = 50\mathbf{i}$ km/hr where $\mathbf{i}$ is a unit vector pointing East.

   The aircraft travels at a constant velocity $\mathbf{u} = (-200\mathbf{i} + 160\mathbf{j})$ km/hr relative to the air. Here $\mathbf{j}$ is a unit vector pointing North.

   (a) Give an expression for the velocity $\mathbf{v}$ of the aircraft relative to the ground.

   (b) Hence write down an expression for the position vector of the aircraft at time t hours.

   (c) Find the time in minutes at which the aircraft has flown 200 km North.

   (d) Find the position vector of the point P the aircraft reaches after it has flown 200 km North.

   [**7 marks**]

7. Evaluate the determinants

$$\begin{vmatrix} 2 & 1 & 2 \\ 3 & -1 & -1 \\ -5 & -1 & 3 \end{vmatrix} \qquad \text{and} \qquad \begin{vmatrix} 3 & 1 & 2 \\ 2 & 2 & 3 \\ 1 & 3 & 4 \end{vmatrix}.$$

   [**4 marks**]

8. Vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are each non-zero, none is parallel to any other, but are such that

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 0\,.$$

   Use the geometrical interpretation of the triple scalar product to deduce what you can about these three vectors, clearly stating your reasons.
   [**4 marks**]

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SECTION B

9. The points A, B, C, D form a parallelogram which acts as the base of a box, whose four vertical faces are rectangular. Side AB is parallel to side DC. The top face is the parallelogram $PQRS$ where the corners P, Q, R and S are adjacent to the corners A, B, C and D respectively. The Cartesian coordinates of A, B, D and P are $(1, 1, 0)$, $(2, 4, 4)$, $(4, 1, 0)$ and $(1, -3, 3)$ respectively.

   (a) Find $\overrightarrow{AB}$, $\overrightarrow{DC}$, $\overrightarrow{AD}$ and $\overrightarrow{BC}$.

   (b) Show that the coordinates of C are (5,4,4).

   (c) Find the vector $\overrightarrow{AP}$, and verify that it is normal to the plane of the base $ABCD$.

   (d) Find the coordinates of Q, R and S.

   (e) Find a vector normal to the plane containing A, C, R and P and hence write down the scalar equation of this plane.

[15 marks]

10. Suppose that the line $\mathcal{L}_1$ has vector equation

$$\mathbf{r} = \mathbf{i} + \mathbf{j} + 2\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j})$$

and that the line $\mathcal{L}_2$ has vector equation

$$\mathbf{r} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k} + \mu(3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k})$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors parallel to the coordinate axes Ox, Oy and Oz.

   (a) Write down the coordinates of *any* two points on the line $\mathcal{L}_1$.

   (b) Determine two unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ which are respectively parallel to the lines $\mathcal{L}_1$ and $\mathcal{L}_2$.

   (c) Show that the angle between the lines is 30 degrees.

   (d) Establish whether the lines do or do not intersect and, if they do, find the coordinates of the point of intersection.

[15 marks]

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11. The planes Π_1 and Π_2 have equations

$$3x - y - z = 2 \qquad \text{and} \qquad 2x - y = 4$$

respectively, with respect to Cartesian axes $Oxyz$.

(a) Find a normal to each plane.

(b) Find the angle in degrees between the normals to the planes Π_1 and Π_2.

(c) Show that the vector equation of the line $\mathcal{L}$ of intersection of the planes Π_1 and Π_2 can be written in the form

$$\mathbf{r} = -4\mathbf{j} + 2\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \ .$$

(d) Find the coordinates of the point A which corresponds to $\lambda = 1$.

(e) Find the Cartesian equation of the plane which is perpendicular to $\mathcal{L}$ and passes through the point A.

[15 marks]

12. The unknowns x, y and z satisfy the simultaneous equations

$$\begin{array}{rcrcrcl} x & - & 3y & + & z & = & 2 \\ 2x & - & 3y & + & 6z & = & 10 \\ 3x & + & 3ay & + & 7z & = & 12 + n \,. \end{array}$$

(a) where a and n are constants. Write down an augmented matrix corresponding to this system of equations.

(b) Use elementary row operations to reduce this augmented matrix to echelon form and show that the entries in the final row are $(0,\, 0,\, -4(a{+}2),\, -6(a + 2) + n)$.

(c) Find a solution for the case with $a = -1$, $n = 0$.

(d) For $a = -2$ what value of n can give a consistent solution? State with reasons whether the solution is unique. Find the solution.

[15 marks]