PAPER CODE NO.
MATH012

THE UNIVERSITY
of LIVERPOOL

SUMMER 2000 EXAMINATIONS

Degree of Bachelor of Science	:	Year 0
Degree of Bachelor of Science	:	Year 1
Degree of Bachelor of Engineering	:	Year 0

VECTORS AND KINEMATICS

TIME ALLOWED : Three Hours

INSTRUCTIONS TO CANDIDATES

Answer ALL questions in Section A and THREE questions from Section B.
The total of the marks available on Section A is 55.

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SECTION A

1. In triangle ABC, the sides AB and BC are given by the vectors $\mathbf{u}$ and $\mathbf{v}$ respectively. The points L and M are the midpoints of the sides BC and AC respectively. Find expressions for the following in terms of $\mathbf{u}$ and $\mathbf{v}$.

   (a) $\overrightarrow{BL}$;

   (b) $\overrightarrow{AM}$;

   (c) $\overrightarrow{ML}$.

[4 marks]

2. The points P, Q and R have Cartesian coordinates $(1, 0, 3)$, $(2, 2, -1)$ and $(3, 1, 1)$ respectively where lengths are measured in centimetres.

   Find

   (a) $\overrightarrow{PQ}$;

   (b) $\overrightarrow{QR}$;

   (c) the coordinates of the point S such that $PQRS$ is a parallelogram with side PQ parallel to side SR ;

   (d) the total length of the sides of parallelogram $PQRS$ in centimetres, to the nearest millimetre ;

   (e) $\overrightarrow{QP} \cdot \overrightarrow{QR}$ and, hence, all angles of the parallelogram $PQRS$, to the nearest degree.

[13 marks]

3. Let $\mathbf{u} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = -\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are mutually orthogonal unit vectors. Find

   (a) $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$;

   (b) $(\mathbf{v} - \mathbf{u}) \cdot \mathbf{v}$ and $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{u}$;

   (c) a unit vector parallel to $\mathbf{u} \times \mathbf{v}$;

   (d) $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v})$.

[10 marks]

4. The straight line $\mathcal{L}$ has vector equation

$$\mathbf{r} = \mathbf{i} - \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors parallel to the x, y and z axes respectively. Here $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

(a) Show that the points B and C, with Cartesian coordinates $(2, 2, 1)$ and $(0, -2, -3)$ respectively, lie on $\mathcal{L}$.

(b) Find, giving arguments, the value of the parameter λ corresponding to the midpoint M of BC.

(c) Find a unit vector parallel to $\mathcal{L}$.

[**7 marks**]

5. Let O be a fixed origin and let $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ be constant, mutually orthogonal unit vectors. A particle P moves so that its position vector $\mathbf{r}$ with respect to O at time t is given by

$$\mathbf{r} = (2t + 1)\mathbf{i} + 4t^2\mathbf{j} + 3te^{-2t}\mathbf{k}$$

where t is measured in seconds and distances are measured in metres. Find:

(a) The position of P at time $t = 0$;

(b) The velocity of P at time t seconds ;

(c) The speed of P at $t = 2$ seconds, to the nearest cm/sec ;

(d) The acceleration of P at $t = 0$.

[**7 marks**]

6. A ferry boat sets out from the origin O to cross a river flowing with constant velocity $\mathbf{w} = 5\mathbf{i}$ km/hr where $\mathbf{i}$ is a unit vector parallel to the river. The ferry boat travels at a constant velocity of $\mathbf{u} = -4\mathbf{i} + 16\mathbf{j}$ km/hr relative to the river. Here $\mathbf{j}$ is a unit vector orthogonal to the river flow.

(a) Give an expression for the velocity $\mathbf{v}$ of the ferry relative to land.

(b) Hence write down an expression for the position vector of the ferry at time t hours.

(c) If the river is 2 km wide, find the time in minutes at which the ferry reaches the opposite side.

(d) Find the position vector of the point P at which ferry reaches the opposite side, giving distances to the nearest metre.

[**7 marks**]

7. Find the value, or values, of x for which the determinant of the matrix

$$\begin{pmatrix} 2 & 0 & x \\ 1 & 1 & -1 \\ x & 4 & -2 \end{pmatrix}$$

is zero.

[**4 marks**]

8. Vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are each non-zero, none is parallel to any other, but are such that

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = 0 \, .$$

Use the geometrical interpretation of the triple scalar product to deduce what you can about these three vectors.

[**4 marks**]

SECTION B

9. The points A, B, C, D form the rectangular base of a box, whose six faces are all rectangles. Side AB is parallel to side DC. The top face is the rectangle $PQRS$ where the corners P, Q, R and S are adjacent to the corners A, B, C and D respectively. The Cartesian coordinates of A, B, D and P are $(1, 0, 1)$, $(2, -2, 0)$, $(3, 1, 1)$ and $(2, -2, 6)$ respectively.

   (a) Find $\overrightarrow{AB}$, $\overrightarrow{DC}$, $\overrightarrow{AD}$ and $\overrightarrow{BC}$.

   (b) Find the coordinates of C.

   (c) Find the vector $\overrightarrow{AP}$, and verify that it is normal to the plane of the base $ABCD$.

   (d) Find the coordinates of Q, R and S.

   (e) Find the volume of the box.

   (f) Write down the vector equation of the diagonal PC.

   (g) Find a vector normal to the plane containing A, C, R and P and hence write down the scalar equation of this plane.

[**15 marks**]

10. The vectors

$$\mathbf{n}_1 = \mathbf{i} + 2\mathbf{k} \quad \text{and} \quad \mathbf{n}_2 = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$$

are respectively normal to the planes Π_1 and Π_2. Here $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the usual unit vectors parallel to the x, y and z axes respectively. The points P and Q with position vectors

$$\mathbf{p} = \mathbf{i} + \mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{q} = 2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$$

lie on Π_1 and Π_2 respectively.

(a) Show that the scalar equations of Π_1 and Π_2 can respectively be written as

$$x + 2z = 3 \quad \text{and} \quad 2x - y + 3z = 15\,.$$

(b) Find the acute angle between the planes Π_1 and Π_2 to the nearest degree.

(c) By solving the scalar equations of the planes Π_1 and Π_2, find the vector equation of their line of intersection.

(d) Verify that the direction of this line is perpendicular to $\mathbf{n}_1$ and $\mathbf{n}_2$.

(e) Give the coordinates of *any* two points which lie on the line of intersection of Π_1 and Π_2 and verify that each does indeed satisfy the equation of each plane.

[**15 marks**]

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11. An aircraft takes off at noon from the origin O and flies such that its position vector t minutes after noon is given by

$$\mathbf{r} = 8t\mathbf{i} + 8t\mathbf{j} + 10\mathbf{k} - e^{-t/5}(6t\mathbf{i} + 2t\mathbf{j} + 10\mathbf{k})$$

where distances are measured in km. The unit vectors $\mathbf{i}$ and $\mathbf{j}$ point due East and North respectively and $\mathbf{k}$ points vertically upwards.

(a) Find the velocity of the aircraft at t minutes after noon.

(b) Find the aircraft's speed at take-off in km per hour and its direction of travel to the nearest degree East of North.

(c) Estimate the final cruising speed of the aircraft i.e. its speed after a sufficiently long time that one can ignore $e^{-t/5}$ in comparison with one.

(d) Estimate the highest altitude reached by the aircraft.

(e) A second, identical aircraft, takes off 2 minutes after the first and travels along the same path in an otherwise identical manner. Write down an expression for its position vector at time t minutes after noon which is valid for $t \geq 2$ minutes.

(f) Find the distance between the two aircraft at 1 pm to the nearest km.

[15 marks]

12. The unknowns x, y and z satisfy the simultaneous equations

$$\begin{array}{rcrcrcr} 4x & - & 6y & + & 10z & = & 13 \\ x & - & 2y & + & 2z & = & 2 \\ 8x & + & 12ay & + & 4z & = & -3\,. \end{array}$$

where a is some fixed parameter.

(a) Write down an augmented matrix for these equations.

(b) Use elementary row operations to reduce this augmented matrix to echelon form.

(c) Find a solution to the above equations in the case $a = -1/2$.

(d) Verify your solution by direct substitution.

(e) Describe the nature of the solutions, if any, in the case where $a = -7/3$.

(f) Deduce the range(s) of values of a for which the equations have a unique solution in x, y and z.

[15 marks]