PAPER CODE NO.
2MA06

THE UNIVERSITY
of LIVERPOOL

SUMMER 1999 EXAMINATIONS

Degree of Bachelor of Science : Year 0
Degree of Bachelor of Science : Year 1
Degree of Bachelor of Engineering : Year 0

VECTORS AND KINEMATICS

TIME ALLOWED : Three Hours

INSTRUCTIONS TO CANDIDATES

Answer ALL questions in Section A and THREE questions from Section B.
The total of the marks available on Section A is 55.

SECTION A

1. Let $ABCD$ be a parallelogram. Given that $\overrightarrow{AB} = \mathbf{u}$ and $\overrightarrow{BC} = \mathbf{v}$, express each of the following in terms of $\mathbf{u}$ and $\mathbf{v}$

   (a) $\overrightarrow{AD}$;

   (b) $\overrightarrow{AC}$;

   (c) $\overrightarrow{BP}$, where P is the point with $\overrightarrow{PC} = 2\overrightarrow{AB}$;

   (d) $\overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{DC}$.

[6 marks]

2. The points P, Q and R have Cartesian coordinates $(0, 2, -1)$, $(1, 3, -3)$ and $(1, 1, -1)$ respectively where lengths are measured in centimetres. Find

   (a) the lengths of the sides of triangle PQR, correct to the nearest millimetre;

   (b) the angles of the triangle PQR in degrees.

[10 marks]

3. Let $\mathbf{a} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = 5\mathbf{j} - 12\mathbf{k}$ where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are mutually orthogonal unit vectors. Find

   (i) $|\mathbf{a}|$ and $|\mathbf{b}|$;

   (ii) $(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$;

   (iii) a unit vector in the direction of $-\mathbf{b}$;

   (iv) $\mathbf{a} \times \mathbf{b}$.

[9 marks]

4. The points A and B have Cartesian coordinates $(1, -1, 2)$ and $(2, -4, 3)$ respectively. Find

   (a) $\overrightarrow{AB}$;

   (b) the coordinates of the midpoint of AB;

   (c) the vector equation of the line through A and B.

[6 marks]

5. A river flows with velocity $\mathbf{u} = 3\mathbf{i}$ km/hour, where $\mathbf{i}$ is a unit vector pointing due East. A ferry sets out to cross the river with velocity $\mathbf{v} = 10\mathbf{j}$ km/hour relative to the river. Here, $\mathbf{j}$ is a unit vector pointing due North. Find

   (a) the speed of the ship relative to the land;

   (b) in what direction the ship is travelling relative to land (degrees East of North).

[8 marks]

6. Let O be a fixed origin and let $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ be constant mutually orthogonal unit vectors. The position vector with respect to O of a particle P is

$$\mathbf{r}(t) = \{2\mathbf{i} + (3t+1)\mathbf{j} + (5t - t^2)\mathbf{k}\}\text{metres.}$$

   at time t seconds. Find

   (a) the position of P at time $t = 0$;

   (b) the velocity of P at time t seconds;

   (c) the speed of P when $t = 2$;

   (d) the acceleration of P at time t seconds.

[8 marks]

7. Evaluate the determinant

$$\begin{vmatrix} 1 & x & 3 \\ x & -1 & 2 \\ 3 & 2 & 4 \end{vmatrix}$$

   where x is some unknown. Find the value or values of x for which the determinant is zero.

[8 marks]

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SECTION B

8. The four distinct points A, B, C and D are non-collinear and such that $\overrightarrow{AB} = \mathbf{u}$, $\overrightarrow{BC} = \mathbf{v}$ and $\overrightarrow{CD} = \mathbf{w}$.

   (a) Find an expression for $\overrightarrow{DA}$ in terms of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

   (b) What condition must be satisfied by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in order that $ABCD$ should be a parallelogram with AB and DC as opposite sides?

   (c) Suppose that, in terms of mutually orthogonal unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$,

   $$\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}, \qquad \mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}, \qquad \text{and} \qquad \mathbf{w} = -2\mathbf{i} + 3\mathbf{j}.$$

      (i) Show that $ABCD$ is *not* a parallelogram.

      (ii) Show that A, B, C and D lie in the same plane.

      (iii) Find a unit vector normal to this plane.

[15 marks]

9. Suppose that the line $\mathcal{L}_1$ has vector equation

   $$\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + 2\lambda\mathbf{j}$$

   and that the line $\mathcal{L}_2$ has vector equation

   $$\mathbf{r} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k} + \mu(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$$

   where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors parallel to the coordinate axes Ox, Oy and Oz .

   (a) Write down the coordinates of *any* two points on the line $\mathcal{L}_1$.

   (b) Write down two vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ which are respectivlely parallel to the lines $\mathcal{L}_1$ and $\mathcal{L}_2$.

   (c) Show that the angle between the direction of the lines is approximately 66 degrees.

   (d) Establish whether the lines do or do not intersect and, if they do, find the coordinates of the point of intersection.

[15 marks]

10. The planes Π_1 and Π_2 have equations

$$2x - y + 2z = 4 \qquad \text{and} \qquad x + 2y - z = 1,$$

respectively, with respect to the coordinate axes Ox, Oy and Oz. The unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are respectively parallel to these axes.

(a) Obtain, in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, unit vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ respectively normal to planes Π_1 and Π_2.

(b) Find the angle between the planes Π_1 and Π_2, correct to the nearest degree.

(c) Show that the line $\mathcal{L}$ of intersection of the planes Π_1 and Π_2 is parallel (or anti-parallel) to the vector $3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k}$.

(d) Obtain the vector equation of $\mathcal{L}$ in terms of some parameter λ.

[**15 marks**]

11. The unknowns w, x, y and z satisfy the simultaneous equations

$$\begin{array}{rcrcrcrcr} 2w & - & x & + & 3y & + & 4z & = & 9 \\ w & & & - & 2y & + & 7z & = & 11 \\ 3w & - & 3x & + & y & + & 5z & = & 8 \\ 2w & + & x & + & 4y & + & 4z & = & 10\,. \end{array}$$

(a) Write down an augmented matrix corresponding to this system of equations.

(b) Use elementary row operations to reduce this augmented matrix to echelon form.

(c) Deduce that the system of equations has a unique solution and find this solution.

(d) Verify your solution by direct substitution in the original equations.

[**15 marks**]