1. (a) In how many ways can you make an **unordered** selection of four integers from the list $\{-4, -3, -2, -1, 1, 2, 3, 4, 5\}$? In how many ways if the integers are distinct?

For how many unordered selections is the product of the four integers positive? How many if the numbers are distinct and their product is positive?

(b) Let S be a set of 5 distinct positive integers each of which is ≤ 9. Prove that the sums of the elements in the subsets of S cannot all be distinct.

2. (a) Obtain a formula for the number of ways of distributing r identical objects into n distinct containers.

(b) How many terms are there in the expansion of $(v + w + x + y)^6$? Give the coefficient of v^2x^3y in this expansion.

(c) For how many sets of non-negative integers $(x_1, x_2, x_3, x_4, x_5)$ do the two equations

$$x_1 + x_2 + x_3 = 5 \text{ and } x_1 + x_2 + x_3 + x_4 + x_5 = 12$$

both hold? How many are there if the equations are replaced by inequalities

$$x_1 + x_2 + x_3 \leq 5 \text{ and } x_1 + x_2 + x_3 + x_4 + x_5 \leq 12?$$

(d) Determine the number of integer solutions of $p + q + r + s = 40$ when:

(i) each of p, q, r, s is non-negative;

(ii) $2 \leq p \leq 19$, $3 \leq q \leq 19$, $4 \leq r \leq 18$ and $3 \leq s \leq 17$.

3. State Philip Hall's Assignment Theorem.

(a) A cube shaped box has sides of length 3. You are given a supply of blocks of size $2 \times 1 \times 1$. Using an alternately black and white colouring of the 27 unit cubes into which the box can be divided, construct a family of sets such that choices of distinct representatives correspond to ways of packing 13 blocks into the box, leaving a blank cube in the middle of the top face, and hence show that such a packing is possible.

Determine also whether or not it is possible to pack the blocks leaving (i) a corner cube or (ii) the middle cube empty.

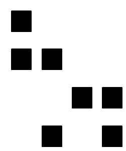
(b) There are 9 applicants for 9 jobs at a firm, and the skills of the applicants are such that those suitable for the respective jobs are:

Job	*Applicants*	*Job*	*Applicants*	*Job*	*Applicants*
1	A, B, C, F	2	A, B, D, E, F	3	A, D, J
4	B, C, D, E, H	5	A, D, G	6	E, F, G, H
7	D, G	8	A, G, J	9	D, J

The Personnel department of the firm allocates the jobs numbered 1-6, 8 to the applicants A, B, D, C, G, E, J in that order. The Managing Director complains at this poor use of manpower, and tells Personnel to reallocate so as to get all the jobs done. Can they do so? Show how to fill the largest possible number of jobs with the minimum of disruption.

4. Define a *rook polynomial.* Give rules which will enable the rook polynomial of any board to be calculated. State the 'forbidden positions' formula.

Obtain the rook polynomials of the boards



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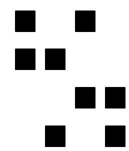

(Here the board consists of the squares indicated by ■.)

If the first two rows of a Latin square are:

1 2 3 4 5 6
2 6 4 3 1 5,

how many possibilities are there for the third row?

5. Solve the following recurrence relations. In each case you should find an expression for a_n and also for the generating function $A(t) = \sum_{n=0}^{\infty} a_n t^n$.

(i) $a_{k+1} = 5a_k - 6a_{k-1}$, $a_0 = 1, a_1 = 4$.

(ii) $a_{k+1} = 2a_k + 2^k$, $a_0 = 1$.

(iii) $a_{k+3} = 6a_{k+2} - 11a_{k+1} + 6a_k$, and $a_0 = a_1 = 1$, $a_2 = 3$.

(iv) $a_{k+1} = 2a_k + (k+4)3^k$, $a_0 = 1$.

(v) $a_{k+1} = 5a_k - 6a_{k-1} + 2.3^k$, $a_0 = 1, a_1 = 7$.

6. (a) State the inclusion-exclusion principle. Obtain a formula for the number D_n of derangements of n objects.

Obtain also a formula for the number of arrangements of $1, 2, \ldots, n$ in a row such that none of the pairs $12, 23, 34, \ldots, (n-1)n$ appear consecutively, and hence show that this number is equal to $D_{n-1} + D_n$.

(b) Find a recurrence relation for the number of ways to cover a $2 \times n$ chessboard with 1×2 dominoes. Find an explicit formula for this number.

7. Establish a formula for the generating function for the number s_n of solutions of the equation $n = a + b + 2c$ in whole numbers a, b and c.

Establish also formulae for the generating function $P^d(x)$ which enumerates the number of partitions of the positive integer n with distinct parts and for the generating function $P_m(x)$ for partitions with each part $\leq m$.

Show that the number of partitions of n with each part $\leq m$ is equal to the number of partitions of n with at most m parts.

Show that the number of partitions of a positive integer n into at most m parts is equal to the number of partitions of $n + \frac{1}{2}m(m+1)$ into m distinct parts.

Hence or otherwise, list all the partitions of 12 into distinct parts.

By obtaining formulae for the generating functions, deduce that

$$\prod_{r=1}^{\infty}(1+t^r) = \sum_{m=0}^{\infty} \frac{t^{\frac{1}{2}m(m+1)}}{(1-t)\dots(1-t^m)}.$$

8. Define the term *symmetric function.* Define the *elementary symmetric functions* and the *power sum symmetric functions* π_n. State and prove the Newton Identities. Hence obtain a formula for π_n in terms of the elementary symmetric functions, and evaluate the formula when $n = 3$.

For any partition λ, define the Schur function S_λ. Express $S_{(1)}, S_{(2)}$ and $S_{(1,1)}$ in terms of the elementary symmetric functions.