1. (a) How many sequences of length 5 can you form from the alphabet

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, +, -, \times, =\}?$$

How many have all five of the entries different? How many of these contain $=$ just once? How many (still with all entries different) have just three of the entries as digits?

How many sequences of length 5 from the above alphabet have $+$ second, $=$ fourth, and digits (not necessarily distinct) in the other three places? And how many of these represent correct sums?

(b) A square has sides of length (a whole number) N. Show that if more than N^2 points are placed within the hexagon, there must be two at distance at most $\sqrt{2}$ apart.

2. Obtain a formula for the number of ways of distributing r identical objects into n distinct containers.

How many terms are there in the expansion of $(v + w + x + y + z)^5$?

Determine the number of integer solutions of $p + q + r + s = 45$ when:

(a) each of p, q, r, s is non-negative;

(b) each of p, q is ≥ 6 and r, s are ≥ 9;

(c) each of p, q, r, s is non-negative and less than 13;

(d) $2 \leq p \leq 19$, $3 \leq q \leq 20$, $4 \leq r \leq 18$ and $6 \leq s \leq 20$.

3. State Philip Hall's Assignment Theorem.

(a) Construct a family of sets which has a system of distinct representatives if and only if the pruned chessboard shown below has a perfect cover. Decide whether the chessboard has a perfect cover or not.

□ □ □ ■ ■
■ □ □ ■ □
■ ■ □ □ □
□ ■ □ □ □
□ □ □ ■ ■

(Here a ■ denotes a square that has been deleted.)

Does it make any difference if the ■ and the □ in the first and second positions in the second row are interchanged?

(b) At a music awards ceremony 8 of the distinguished guests arrive late, and the 8 vacant seats are all at different tables. Because of personality clashes, the only guests prepared to sit at Table 1 are A,B,C,E; at Table 2 are A,B,D; at Table 3 A,D; at Table 4 B,C,D,E,G; at Table 5 B,D,F; at Table 6 A,B,F; at Table 7 E,F,G,H; and at Table 8 D,F. Can the head waiter seat them all? Show how he can seat the maximum number possible, and justify your answer.

4. Define a *rook polynomial.* Give rules which will enable the rook polynomial of any board to be calculated. State the 'forbidden positions' formula.

If the first row of a Latin square is (1,2,3,4,5,6), how many possibilities are there for the second row? If the first two rows are:

1 2 3 4 5 6
2 3 4 5 6 1,

how many possibilities are there for the third row?

5. Solve the following recurrence relations. In each case you should find an expression for a_n and for the generating function $A(t) = \sum_{n=0}^{\infty} a_n t^n$.

(i) $a_{k+3} = 6a_{k+2} - 11a_{k+1} + 6a_k$, and $a_0 = a_1 = 1$, $a_2 = 3$.

(ii) $a_{k+1} = 4a_k - 4a_{k-1}$, $a_0 = a_1 = 1$.

(iii) $a_{k+1} = 4a_k - 4a_{k-1} + 1$, $a_0 = a_1 = 1$.

(iv) $a_k = 4a_{k-1} + 3k$, $a_0 = 0$.

(v) $a_k = 3a_{k-1} + 2^k$, $a_0 = 1$.

(vi) $a_k = 3a_{k-1} + 3^k$, $a_0 = 1$.

6. (a) State the inclusion-exclusion principle. Hence obtain a formula for the number D_n of derangements of n objects. Calculate these numbers for $n \leq 6$. Find a formula for the number of permutations of n objects which leave just r of the objects unmoved. Calculate these numbers for $n = 6$ and $0 \leq r \leq 6$, and check that they add up to 6!.

(b) For $n \geq 0$ suppose given $2n$ points on the circumference of a circle, and labelled clockwise with the integers $1, 2, \ldots, 2n$ in order. Let c_n be the number of ways in which these points can be paired off by n chords of the circle with no two chords intersecting and no two chords joining the same vertex. Sketch the possibilities for $n = 3$. Show that there is a chord joining the first vertex to a vertex whose number is an even integer. Obtain the recurrence relation $c_n = \sum_{i=1}^{n} c_{i-1} c_{n-i-1}$, and hence a quadratic equation satisfied by the generating function $C(x) = \sum_{n=0}^{\infty} c_n x^n$. Solve this and deduce that $c_n = \frac{(2n)!}{n!(n+1)!}$.

7. Establish formulae for the generating function $P(x)$ which enumerates the number $p(n)$ of partitions of the positive integer n, and the generating functions $P_o(x)$ for the number of partitions into odd parts, and $P_d(x)$ for partitions with distinct parts. For any positive integer n, show that the number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts.

Show that the number of partitions of a positive integer n into parts none of which occurs more than twice is equal to the number of partitions of n where no part is divisible by 3.

Now let $p_1(n,k)$ denote the number of partitions of $2n+k$ with one of the parts equal to $n+k$, and let $p_2(n,k)$ denote the number of partitions of $2n+k$ with precisely $n+k$ parts. Use Ferrers graphs to show that $p_1(n,k) = p_2(n,k)$ and that $p_1(n,k) = p(n)$. Deduce that the number of partitions of $2n+k$ into precisely $n+k$ parts is independent of k.

8. Define the term *symmetric function.* For any positive integer n, define the *elementary symmetric function* σ_n and the *power sum symmetric function* π_n. State and prove the Newton Identities.

The *homogeneous product sum* h_n is the sum of all monomials in the variables of total degree n. Obtain a formula for the generating function $\sum_{n=0}^{\infty} h_n t^n$, and hence obtain a relation between the h_n and the σ_n. By using the generating functions, obtain also a relation between the h_n and the π_n.

Express each of h_1, h_2 and h_3 in terms of the elementary symmetric functions and also in terms of the power sum functions.