1. (a) The government of Ruritania has recently established a National Lottery. Each ticket carries six numbers, all different, which may be chosen between 1 and 30 inclusive, of which four are coloured white and two are coloured red. How many possible such choices are there?

Mrs. Grundy regards numbers divisible by 3 as lucky, and only chooses such numbers when she buys a ticket. In how many ways can she choose three (different) tickets?

How much does her superstition affect her chance of winning?

(b) Let S be a set of 6 distinct positive integers each of which is ≤ 13. Prove that the sums of the elements in the subsets of S cannot all be distinct.

2. Obtain a formula for the number of ways of distributing r identical objects into n distinct containers.

Determine the number of integer solutions of $p + q + r + s = 38$ when:

(a) each of p, q, r, s is non-negative;

(b) each of p, q is ≥ 5 and r, s are ≥ 9;

(c) each of p, q, r, s is ≥ -2;

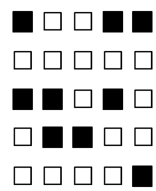
(d) each of p, q, r, s is non-negative and less than 10;

(e) each of p, q, r, s is non-negative and less than 13;

(f) $0 \leq p \leq 16$, $0 \leq q \leq 17$, $3 \leq r \leq 20$ and $5 \leq s \leq 20$.

3. State Philip Hall's Assignment Theorem.

(a) Construct a family of sets which has a system of distinct representatives if and only if the pruned chessboard shown below has a perfect cover. Decide whether the chessboard has a perfect cover or not.



(Here a ■ denotes a square that has been deleted.)

(b) Prof. A is making up a lecture schedule for his department: he has 6 further members of staff (Dr. B, Dr. C, Dr. D, Dr. E, Dr. F and Dr. G) and 7 unit courses $2DM1, \cdots, 2DM7$. The courses which $A, \cdots G$ are competent to teach are, respectively, those numbered $\{1{,}3\}$, $\{2{,}6\}$, $\{2{,}3{,}4\}$, $\{1{,}4{,}5\}$, $\{3{,}4{,}5\}$, $\{4{,}6\}$ and $\{4{,}7\}$, and he allocates $A, \cdots, G$ to teach courses 2DM1 to 2DM7 in that order.

At the last minute, the Department acquires a temporary member of staff (Mr. H) who is anxious to teach a course but can only teach $\{1{,}5{,}6\}$. Dr. B proposes that he will offer a new course 2DM8 instead of 2DM2. How can Prof. A reallocate duties with a minimum of disruption?

(To obtain full credit, you must show why no other choice requires fewer changes in the schedule.)

4. Define a *rook polynomial.* Give rules which will enable the rook polynomial of any board to be calculated. State the 'forbidden positions' formula. Calculate the rook polynomial for a 3×3 chessboard with the 3 diagonal squares removed.

The first two rows of a Latin square are:

$$\begin{array}{c} 2\,3\,1\,5\,6\,4 \\ 3\,1\,2\,6\,4\,5. \end{array}$$

How many possibilities are there for the next row?

5. (a) Solve the following recurrence relations. In each case you are given $a_0 = 1,\ a_1 = 3$, and should find an expression for a_n and for the generating function $A(t) = \sum_{n=0}^{\infty} a_n t^n$.

(i) $a_{k+1} = 5a_k - 6a_{k-1}$,

(ii) $a_{k+1} = 4a_k - 4a_{k-1}$,

(iii) $a_{k+1} = 2a_k - a_{k-1} + 1$.

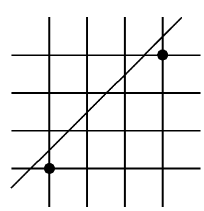
(b) Let Δ_n be the determinant of the $n \times n$ matrix with 1's in the diagonals immediately above and immediately below the main diagonal and 0's elsewhere, so that, for example,

$$\Delta_5 = \det \begin{pmatrix} 0\,1\,0\,0\,0 \\ 1\,0\,1\,0\,0 \\ 0\,1\,0\,1\,0 \\ 0\,0\,1\,0\,1 \\ 0\,0\,0\,1\,0 \end{pmatrix}.$$

Find a linear recursion formula for Δ_n, and hence show that $\Delta_n = \sin \frac{1}{2} n\pi$.

6. (a) State the inclusion-exclusion principle. Hence obtain a formula for the number D_n of derangements of n objects.

(b) A small town has streets laid out in a square grid, with a railway line running diagonally. Two pubs are situated on street corners by the railway line (as illustrated for $n = 3$), and are n blocks apart in an E-W direction and also in an N-S direction. Write c_n for the number of routes of shortest distance ($2n$ blocks) from one pub to the other that do not cross the railway line. Calculate c_3.



Find a recurrence relation for c_n, and deduce that c_n is the n^{th} Catalan number.

7. Obtain a formula for the generating function $S(x) = \sum_{n=1}^{\infty} s_n x^n$, where s_n is the number of solutions of $n = 2a + 3b + 6c$ in non-negative integers.

Establish also the generating functions $p(x)$, which enumerates the number of partitions of n and $p^m(x)$ for the partitions of n with each part $\leq m$.

Define Ferrers graphs, and use them to explain why the generating function $p_m(x)$ for the number of partitions of n into at most m parts satisfies $p_m(x) = p^m(x)$.

Define the Durfee square, and give the size of the Durfee square of the partition $\{10, 8, 6, 4, 2\}$?

Using Durfee squares, show that the function

$$\sum_{m=1}^{\infty} \frac{x^{m^2}}{\prod_{i=1}^{m}(1-x^i)^2}$$

enumerates partitions, and hence is equal to $p(x)$.

8. Define the term *symmetric function*. For any positive integer n, define the *elementary symmetric function* σ_n and the *power sum symmetric function* π_n. State the Newton Identities.

Express each of the following (a) in terms of the power sum functions (of α, β, γ), and (b) in terms of their elementary symmetric functions:

(i) $\alpha^3 + \beta^3 + \gamma^3$,

(ii) $\alpha^2(\beta+\gamma) + \beta^2(\gamma+\alpha) + \gamma^2(\alpha+\beta)$,

(iii) $\begin{vmatrix} 1 & 1 & 1 \\ \alpha^2 & \beta^2 & \gamma^2 \\ \alpha^3 & \beta^3 & \gamma^3 \end{vmatrix} \Big/ \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix}$.