1. (a) A club has 28 members: 16 men and 12 women. In how many ways can they elect a committee of 6 members? In how many ways can they do it if a rule is adopted that there must be at least 2 of each sex on the committee?

An additional change is agreed to the rules, specifying that the society elect annually a Chairperson, a Secretary, a Treasurer and 3 further committee members: in how many ways can this be done?

An amendment to this rule requires that there must be at least one officer of each sex. How many possibilities are there now?

(b) Let T be a circular disc of radius 1 cm. Show that if 7 points are marked on T, there must be two within a distance of 1 cm. of each other.

2. Obtain a formula for the number of ways of distributing r identical objects into n distinct containers.

Determine the number of integer solutions of $p + q + r + s = 40$ when:

(a) each of p, q, r, s is non-negative;

(b) each of p, q is ≥ 5 and r, s are ≥ 10;

(c) each of p, q, r, s is ≥ -1;

(d) each of p, q, r, s is non-negative and less than 9;

(e) each of p, q, r, s is non-negative and less than 13;

(f) $0 \leq p \leq 16$, $0 \leq q \leq 18$, $3 \leq r \leq 20$ and $5 \leq s \leq 22$.

3. State Philip Hall's Assignment Theorem.

(a) Construct a family of sets which has a system of distinct representatives if and only if the pruned chessboard shown below has a perfect cover. Decide whether the chessboard has a perfect cover or not.

□ ■ □ □ □
□ □ □ ■ ■
■ ■ □ □ □
□ □ □ ■ □
■ □ ■ ■ □

(Here a ■ denotes a square that has been deleted.)

(b) A works manager has 8 employees to assign to 6 jobs. Those competent to do the jobs are, respectively, $\{A, E\}$, $\{B, C, E\}$, $\{C, H\}$, $\{D, G\}$, $\{C, E, F\}$ and $\{C, D\}$, and he assigns these jobs to A, B, C, G, E and D respectively.

To maintain full employment he finds a further job, for which just A, B and C are competent: how can he reassign the work schedule with minimum disruption? One more job is found, for which just A, B and H are competent. Can he reschedule the jobs so that each man has one, and if so, how should he do it?

4. Define a *rook polynomial* and calculate the rook polynomial for a 3×3 chessboard. Give rules which will enable the rook polynomial of any board to be calculated. State the 'forbidden positions' formula.

The first two rows of a Latin square are:

$$\begin{matrix} 1\,2\,3\,4\,5\,6 \\ 2\,1\,4\,3\,6\,5. \end{matrix}$$

How many possibilities are there for the next row?

5. (a) Solve the following recurrence relations. In each case you are given $a_0 = 1$, $a_1 = 4$, and should find an expression for a_n and for the generating function $A(t) = \sum_{n=1}^{\infty} a_n t^n$.

(i) $a_{k+1} = 2a_k - a_{k-1}$,

(ii) $a_{k+1} = 3a_k - 2a_{k-1}$,

(iii) $a_{k+1} = 3a_k - 2a_{k-1} + 1$.

(b) Write $A(n, k)$ for the number of pieces into which $\mathbb{R}^n$ is cut by k general linear hyperplanes. Then $A(n, k) = 1$ when $n = 0$ or $k = 0$, and $A(n, k)$ satisfies the recurrence relation

$$A(n, k) - A(n, k-1) = A(n-1, k-1).$$

Find an expression for the generating function $C(k) = \sum_{n \geq 0} A(n, k) y^n$. Hence obtain a simplified expression for

$$B(x, y) = \sum_{k \geq 0,\, n \geq 0} A(n, k) x^k y^n.$$

6. (a) Let $a_n = \int_0^1 x^n e^x dx$. Obtain a recurrence relation by integrating by parts.

Write down a formula for the number D_n of derangements of n objects. Prove by induction that $a_n = (-1)^n (eD_n - n!)$.

(b) Let c_n be the number of ways to arrange n symbols '(' and n symbols ')' in a row so that the bracketing is consistent: i.e. at each point you must have opened at least as many parentheses as you have closed. Compute c_3 by listing all cases.

Show that in any consistent arrangement there is a unique way to pair symbols '(' with symbols ')' so that in each pair (comes before). Show also that two paired symbols have an even number of symbols between them.

By considering which) is paired with the first (obtain a recurrence relation for c_n. Hence or otherwise show that c_n is the nth Catalan number.

7. Establish the generating function $p(x)$ which enumerates the number of partitions of the positive integer n. Find also the function $p_o(n)$ which enumerates the number of partitions of n into odd parts, and the function $p_d(x)$ to enumerate the partitions of n with distinct parts. For any positive integer n, show that the number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts.

Define Ferrers graphs. Use these to establish a bijection between self-conjugate partitions of n and partitions with distinct odd parts. Write down the generating function which enumerates such partitions, and hence determine the number of self-conjugate partitions of 13. Exhibit the corresponding Ferrers graphs.

8. Define the term *symmetric function.* For any positive integer n, define the *elementary symmetric function* σ_n and the *power sum symmetric function* π_n. State and prove the Newton Identities.

Express σ_3 in terms of the power sum functions and π_4 in terms of the elementary symmetric functions.

Obtain, in terms of the elementary symmetric functions of α, β and γ, the equation with roots $\alpha + \beta,\ \beta + \gamma,\ \gamma + \alpha$.

Write δ_r for the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^r & \beta^r & \gamma^r \end{vmatrix}$$

Show that, for each $r \geq 3$, $\phi_r = \delta_r/\delta_2$ is a symmetric function of α, β and γ. Express ϕ_4 in terms of the elementary symmetric functions.