2MP73: Curves and Singularities
Draft Examination 1997

1. (i) Define the *distance-squared function* f from the point (a, b) to a plane curve $\gamma : I \to \mathbf{R}^2$. State the condition, in terms of f, for γ to have a vertex at $t = t_0$, and the condition for γ to have a higher vertex at $t = t_0$.

Let γ be given by $\gamma(t) = (t^2, t + t^4)$ (here, $I = \mathbf{R}$). Show that γ is a regular curve for all t. Write down the distance-squared function from (a, b) to γ, and, hence or otherwise, show that γ has a vertex, but not a higher vertex, at $t = 0$. Find the centre of curvature at this point.

Define the *height function* h on a plane curve γ in the direction $u = (a, b)$ (where a, b are not both zero). Let $\gamma(t) = (t^2, g(t))$, where g is smooth and $g'(0) \neq 0$. Show that γ is a regular curve and that γ has an inflexion at t if and only if $tg''(t) = g'(t)$. What is the condition on g that γ has a higher inflexion? For $g(t) = t + t^4$, find the inflexion point or points on γ, and determine for each whether it is a higher inflexion.

2. (i) Let $\alpha : I \to \mathbf{R}^3$ be a unit speed space curve. Define T and κ, and, assuming $\kappa \neq 0$, define N, B and τ. Show that $B' = -\tau N$. Express the third derivative of α as a linear combination of T, N and B and show that

$$\tau = \frac{[\alpha', \alpha'', \alpha''']}{||\alpha''||^2}.$$

Verify that $\alpha(s) = (\frac{3}{5}\sin s, \frac{4}{5}s, \frac{3}{5}\cos s)$ is unit speed. Find T, κ, N, B and τ for this curve.

(ii) Explain briefly how to measure contact between a space curve and a plane. Find the order of contact between the curve

$$\gamma(t) = (t, t^2, t^4 - 2t^5 + t^6)$$

and the plane $z = 0$, at each point where they meet.

3. (i) Let $f : \mathbf{R}, t_0 \to \mathbf{R}$ be smooth. Write down what it means to say that f has an A_k singularity at t_0.

Let $f(t) = 3t^4 + 4t^3 + 1$. Find all the values of $t = t_0$ for which f has an A_k singularity for some $k \geq 1$ at t_0. For each one, find an explicit local diffeomorphism

$$h : \mathbf{R}, t_0 \to \mathbf{R}, 0 \ \text{ such that } f(t) = f(t_0) \pm (h(t))^{k+1}.$$

(ii) Let $\phi : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $\phi(x, y) = (w, z) = (x, xy - y^4)$. Write down the Jacobian matrix J of ϕ, and sketch in the (x, y)-plane the critical set of ϕ, where $\det J = 0$. Find all points (x, y) where $\phi(x, y) = (1, 0)$. What does the Inverse Function Theorem say about local inverses of ϕ for (w, z) close to $(1, 0)$? For each such local inverse, find $\frac{\partial y}{\partial z}$ at $(w, z) = (1, 0)$.

4. Let $f : \mathbf{R} \to \mathbf{R}$ be a smooth function. Show that there is a global diffeomorphism $\phi : \mathbf{R}^2 \to \mathbf{R}^2$ taking the x-axis to the graph of f, that is, the set of points $(t, f(t))$ in $\mathbf{R}^2$.

Show that each of the formulae

$$\theta(x,y) = (x^3 + y, x^4), \qquad \psi(x,y) = (x + y, x^{\frac{4}{3}})$$

defines a map $\mathbf{R}^2, (0,0) \to \mathbf{R}^2, (0,0)$ taking the x-axis to the curve γ in $\mathbf{R}^2$ parametrized $\gamma(t) = (t^3, t^4)$. Show also that neither θ nor ψ is a local diffeomorphism.

Show that there is *no* local diffeomorphism

$$\phi : \mathbf{R}^2, (0,0) \to \mathbf{R}^2, (0,0)$$

taking the x-axis near (0,0) to the above curve γ in $\mathbf{R}^2$ [Hint: Let $\phi(x,0) = (f(x), g(x))$ and assume $f^4 = g^3$ for all x close to zero. Differentiate this expression with respect to x several times and put $x = 0$.]

5. Define the terms *regular point* and *regular value* as applied to a map $f : \mathbf{R}^m \to \mathbf{R}^q \ (q \le m)$.

(i) Let $f : \mathbf{R}^2 \to \mathbf{R}$ be given by $f(x,y) = y^2 - x^2 + x^3$. Show that 0 is not a regular value of f but that 0 is a regular value of $f_1 = f|(\mathbf{R}^2 - \{(0,0)\})$. What does the Implicit Function Theorem say about $C = f_1^{-1}(0)$? Which of the variables x, y can be used as local parameter on C close to $(1,0)$? Find the curvature of the curve C at $(1,0)$.

(ii) Let $g_1(x,y,z) = x^2 + y^2 - z^2, \ \ g_2(x,y,z) = y^2 + z^2 - 1$ define maps

$$g_1 : \mathbf{R}^3 - \{(0,0,0)\} \to \mathbf{R}, \qquad g_2 : \mathbf{R}^3 \to \mathbf{R}.$$

Show that 0 is a regular value of g_1 and that (0,0) is a regular value of g, defined by $g(x,y,z) = (g_1(x,y,z), g_2(x,y,z))$. What can you conclude about the sets in $\mathbf{R}^3$ given by (a) $g_1 = 0$, (b) $g_1 = g_2 = 0$? Find a tangent vector to the curve $g_1 = g_2 = 0$ at $(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{\sqrt{3}}{2})$.

6. Let $\gamma : I \to \mathbf{R}^2$ be a plane curve with $\gamma(t)$ never equal to (0,0). Show that the foot of the perpendicular from the origin (0,0) to the tangent to γ at t is

$$\delta(t) = (\gamma(t).N(t))N(t),$$

where N is the unit normal to γ. Show that the equation of the circle, centre $\gamma(t)$ and passing through $(0,0)$ is $(\mathbf{x} - 2\gamma(t)).\mathbf{x} = 0$.

(i) You may assume in this part that γ is unit speed. Let $F : I \times \mathbf{R}^2 \to \mathbf{R}$ be defined by

$$F(t,x) = (\mathbf{x} - 2\gamma(t)).\mathbf{x} \ .$$

Show that the envelope of F consists of (0,0) together with the curve $2\delta(t)$. Find the points of regression on the envelope, other than (0,0).

(ii) Let $\gamma(t) = (t-1, t^2)$ (here $I = \mathbf{R}$). Sketch the curve γ, write down $N(t)$ and find a parametrization of δ, $\delta(t) = (X(t), Y(t))$, say. Verify that $X = -2tY$ and show that δ is a regular curve for all t.

7. (i) In each of the following cases, the formula F gives an unfolding of the function $f(t) = F(t, \mathbf{0})$ at $t = 0$. Determine the A_k type of the function f at 0 and whether the unfolding is versal.

$$\begin{aligned} F(t, x, y) &= t^3 + x^2t^2 + (x + y)t + y. \\ F(t, x, y, z) &= t^4 + (x + y)t^2 + x^2t + y + z. \end{aligned}$$

(ii) Let α, β be two unit speed curves, with parameters s, t respectively, and such that $\alpha'(0) = (1, 0), \beta'(0) = (-1, 0)$. Thus the tangents at $\alpha(0), \beta(0)$ are parallel and pointing opposite ways. In what follows, $T_\alpha, N_\alpha, \kappa_\alpha$ refer to the unit tangent, unit normal and curvature of α, and similarly for β. Let

$$f : \mathbf{R} \times \mathbf{R}, (0, 0) \to \mathbf{R} \quad \text{be} \quad F(s, t) = N_\alpha(s).T_\beta(t).$$

Explain why $f(s, t) = 0$ if and only if the tangents at $\alpha(s), \beta(t)$ are parallel. Show that, provided $\kappa_\alpha(0)$ or $\kappa_\beta(0)$ is non-zero, $f^{-1}(0)$ is, close to (0,0), a smooth curve.

Let $m : \mathbf{R} \times \mathbf{R}, (0, 0) \to \mathbf{R}^2$ be $m(s, t) = \frac{1}{2}(\alpha(s) + \beta(t))$. Describe the curve $m(f^{-1}(0))$ and show that it is a regular curve close to $m(0, 0)$ provided $\kappa_\alpha(0) \neq \kappa_\beta(0)$.

8. Let $\gamma : I \to \mathbf{R}^3$ be a unit speed space curve with curvature never zero. The *normal plane* to γ at t is the plane through $\gamma(t)$ orthogonal to $T(t)$, i.e. with equation $F(\mathbf{x}, t) = 0$, where $F(\mathbf{x}, t) = (\mathbf{x} - \gamma).T$ (here, $\mathbf{x} \in \mathbf{R}^3, t \in I$).

Show that the envelope of these normal planes contains one line in each normal plane, having the form

$$\mathbf{x} = \gamma + \frac{1}{\kappa}N + \mu B,$$

where μ is arbitrary.

Show that the points of regression on the envelope are given by either (i) $\tau = \kappa' = 0, \mu$ arbitrary, or (ii) $\tau \neq 0, \mu = \kappa'/(\kappa^2\tau)$.

Find the 1-jet and 2-jet matrices with constants for the unfolding F. Show that the 1-jet matrix always has rank 2 and the 2-jet matrix has rank 3 if and only if $\tau \neq 0$,

What can you deduce about the structure of the envelope of normal planes at $\mathbf{x}_0$, when (i) $F(\mathbf{x}_0, t)$ has type A_2 at t_0, (ii) $F(\mathbf{x}_0, t)$ has type A_3 at t_0? (You need not calculate the conditions for these A_k types to occur.)