2MP44 May 1999

SECTION A

1. Say what it means for $\{v_1, \dots, v_k\}$ to *span* a vector space V.
Let U be the subspace of $\mathbf{R}^3$ spanned by

$$u_1 = (1, 0, -1), u_2 = (1, -2, 1), u_3 = (2, 2, -4).$$

Let W be the subspace of $\mathbf{R}^3$ spanned by

$$w_1 = (2, 1, -3), w_2 = (2, 3, -5), w_3 = (1, -1, 0).$$

Show that $U = W$.

[9 marks]

2. Define the terms: *group, homomorphism, kernel, image.*

Let G be the set of all 3×3 matrices of the form $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbf{R}$, under the operation of matrix multiplication. Let H be the group of real numbers, under the operation of addition [you need not show that these are groups]. Let $\phi : G \to H$ be defined by

$$\phi\left(\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}\right) = a.$$

Show that ϕ is a homomorphism. Find the kernel and image of ϕ.

[10 marks]

3. Let V be the vector space of 2×2 real matrices. Let the map $L : V \to V$ be defined by

$$L\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a+d & c \\ b & a+d \end{pmatrix}.$$

Prove that L is a linear map and compute its rank and nullity.

[9 marks]

4. Calculate the matrix M of the linear map $\phi_{O,\alpha}$ which corresponds to rotation of the plane anti-clockwise through an angle of α about the origin

O. Let σ_ℓ denote the linear map representing the isometry which is reflection of the plane in the line ℓ with equation $x = 0$, and σ_k correspond to reflection of the plane in the line k with equation $x = y$. Write down the matrices A, B of σ_ℓ, σ_k respectively. Compute the matrix C of the composite map $\sigma_\ell\sigma_k$ and decide whether this composite map is itself a reflection or not. Find the smallest positive integer such that C^n is the identity matrix, and interpret this geometrically. [9 marks]

5. Let f be the bilinear form on $\mathbf{R}^2$ defined by

$$f((x_1, x_2), (y_1, y_2)) = x_1y_1 - x_1y_2 + x_2y_2.$$

Let $u_1 = (2, 2), u_2 = (0, 1)$. Compute $f(u_1, u_1), f(u_1, u_2), f(u_2, u_1), f(u_2, u_2)$. Find the matrix A of f relative to the basis $\{u_1, u_2\}$. Find the matrix B of f relative to the basis $\{v_1, v_2\}$, where $v_1 = (1, 1), v_2 = (0, -1)$.

Find the change of basis matrix P from $\{u_1, u_2\}$ to $\{v_1, v_2\}$ and show that $B = P^TAP$. [9 marks]

6. Consider the following three vectors in the space $V = \mathbf{R}^3$,

$$v_1 = (0, 1, -1), \quad v_2 = (1, 2, 3) \quad \text{and} \quad v_3 = (1, -2, 4).$$

Show that v_1, v_2 and v_3 form a basis for V. Let ϕ_1, ϕ_2 and ϕ_3 be the dual basis for $\{v_1, v_2, v_3\}$. Find an expression for the value of each of the maps ϕ_1, ϕ_2, ϕ_3 at a general point (x, y, z) of V.

[9 marks]

SECTION B

7. Let V be the vector space of 2×2 real matrices. Let $f : V \to V$ be the map defined by

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a+b+c+d & a-b-c+d \\ a+b-c-d & a-b+c-d \end{pmatrix}$$

Prove that f is a linear map. Find the matrix of f with respect to the basis

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Find a basis for the image of f and a basis for the kernel of f. Find the rank of f and the nullity of f. State the rank and nullity theorem and check that it holds in this case.

8. Let V be the vector space of polynomials of degree at most 3 with real coefficients . Let $f : V \to V$ be the linear map defined by

$$f(a + bx + cx^2 + dx^3) = a + bx + dx^2 + cx^3.$$

Find A, the matrix of f with respect to the standard basis, $\{1, x, x^2, x^3\}$, of V and the matrix B of f with respect to the basis $\{1, x, x^2 + x^3, x^2 - x^3\}$ of V.

Find the eigenvalues and eigenvectors of A and hence write down the diagonal form D of A. Explain the connection between B and D.

9. Consider the quadratic form

$$q(x, y, z) = x^2 + 6xy + y^2 + 4z^2.$$

Write down the matrix A representing q with respect to the standard basis. Find a diagonal matrix D equivalent to A and the orthogonal matrix P which describes the change of basis from the standard basis to the basis in which q is diagonal. What are the rank and signature of q? Describe geometrically the surface $q(x, y, z) = 25$. Draw a sketch of the surface.

10. (i) Let G be a group. Show that the identity element e is unique.

(ii) Show that, for any $\alpha, \beta, \gamma \in G$, $\alpha * \beta = \alpha * \gamma \Rightarrow \beta = \gamma$. Deduce that no element can be repeated in the same row inside a group table. Similarly show that no element can be repeated in the same column of table.

(iii) Let X be a set with five elements, $\{e, a, b, c, d\}$, with an operation $\circ$ which satisfies the following table:

$\circ$	e	a	b	c	d
e	e	a	b	c	d
a	a	e	c	d	b
b	b	d	e	a	c
c	c	b	d	e	a
d	d	c	a	b	e

Find an example to show that $\circ$ is not an associative operation.

Suppose now that G is a group with five elements $\{e, a, b, c, d\}$ with e being the identity of G, and the elements labelled so that $a \circ b = c$. Fill in the multiplication table to decide whether or not it is possible for the elements of G to satisfy

$$a^2 = b^2 = c^2 = d^2 = e.$$