SECTION A

1. State (without proof) whether or not each of the following sequences (x_n) is i) increasing; ii) decreasing; iii) bounded above; iv) bounded below. State also the supremum, infimum, maximum, and minimum of each sequence for which they exist.

a) $x_n = \frac{(-1)^n n}{n+1}$ $(n \geq 0)$.

b) $x_n = n(n-1)/2$ $(n \geq 0)$. [8 marks]

2. Calculate the value of, and the first four convergents to, each of the following continued fractions:

a) $[1, 1, 1, 1, 1, 1 \ldots]$.

b) $[2, 3, 2, 3, 2, 3, \ldots]$.

(Recall the formulae: $p_0 = a_0$, $p_1 = a_1 a_0 + 1$, $p_n = a_n p_{n-1} + p_{n-2}$ for $n \geq 2$; $q_0 = 1$, $q_1 = a_1$, $q_n = a_n q_{n-1} + q_{n-2}$ for $n \geq 2$).

[8 marks]

3. Let $f_r\colon [0,1] \to [0,1]$ be given by $f_r(x) = rx(1-x)$, where the parameter $r \in [0,4]$. Calculate the fixed points of f_r, and determine the range of values of r for which each is stable. (You are not required to determine the stability of the fixed points when $f_r'(x)$ is ± 1). [7 marks]

4. State without proof whether each of the following sets is open, closed, both, or neither.

a) $[0, 1)$, as a subset of $\mathbf{R}$.

b) $\mathbf{R}$, as a subset of $\mathbf{R}$.

c) $\{(x, y) : 0 < x^2 + y^2 < 1\}$, as a subset of $\mathbf{R}^2$.

d) $\{(x, y) : x + 2y > 3\}$, as a subset of $\mathbf{R}^2$.

[8 marks]

5. A map $f\colon [0,1] \to [0,1]$ is such that f has 4 fixed points, f^2 has 6 fixed points, f^3 has 10 fixed points, and f^4 has 18 fixed points. How many periodic orbits does f have of each of the periods 2, 3, and 4?

[6 marks]

6. Calculate the Fourier series expansion of t^2 $(t \in [-\pi, \pi))$. [10 marks]

7. Sketch the 2π-periodic extensions of each of the following functions $f(t)$, defined for $t \in [-\pi, \pi)$. State whether each is i) continuous, ii) piecewise continuous, iii) differentiable, and iv) piecewise differentiable. Give the values of $f(t^-)$ and $f(t^+)$ at each discontinuity.

a) $f(t) = |t|$.

b) $f(t) = 1 - t$. [8 marks]

SECTION B

8. Using the Newton-Raphson formula, construct an iteratively defined sequence (x_n), starting with $x_0 = 2$, which tends to $\sqrt[3]{5}$ as $n \to \infty$. Prove that the sequence does indeed tend to $\sqrt[3]{5}$. (You may use any results from the lectures without proof, but they should be clearly stated). [15 marks]

9. What is meant by a subsequence of a sequence (x_n)? State the Bolzano-Weierstrass theorem concerning the existence of convergent subsequences of a sequence (x_n) $(x_n \in \mathbf{R})$.

State whether or not each of the following sequences (x_n) has a convergent subsequence. Give reasons for your answers.

a) $x_n = 1/(1+n)$.

b) $x_n = (-1)^n n^2$.

c) $x_n = n$th digit in the decimal expansion of e.

d) $x_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ n^n & \text{if } n \text{ is odd.} \end{cases}$

e) $x_n = n/p_n$, where p_n is the nth prime number.

[15 marks]

10. State Sharkovsky's theorem (concerning the possible sets of periods of continuous maps $f\colon [0,1] \to [0,1]$).

Determine the Markov graphs of the period 4 pattern (1 2 3 4) and the period 6 pattern (1 4 3 5 2 6). Show that a continuous map $f\colon [0,1] \to [0,1]$ with a period 4 orbit of pattern (1 2 3 4) must have periodic orbits of every period. What other periods of orbits must a map with a period 6 orbit of pattern (1 4 3 5 2 6) have?

[15 marks]

11. What does it mean for a square matrix P to be a stochastic matrix? Under what conditions is it true that there is a unique eigenvector $\mathbf{x}$ of P with eigenvalue 1, such that $P^n\mathbf{x}_0 \to \mathbf{x}$ as $n \to \infty$ for all vectors $\mathbf{x}_0$ with non-negative entries adding up to 1?

An examiner is setting a very long exam on Numbers, Iteration, and Fourier Series. If a given question is about Numbers, then the following question will be about Numbers with probability 1/6, Iteration with probability 1/2, and Fourier Series with probability 1/3; if it about Iteration, then the following question will be about Numbers with probability 1/3, Iteration with probability 1/6, and Fourier series with probability 1/2; and if it is about Fourier series, then the following question will be equally likely to be on any of the three topics.

In the long run, what proportion of the questions will be about Fourier Series?

[15 marks]

12. Let the Fourier series expansion of e^t $(t \in [-\pi, \pi))$ be

$$\frac{a_0}{2} + \sum_{r=1}^{\infty}(a_r \cos rt + b_r \sin rt).$$

a) Show (using the usual formula for calculating a_0) that $a_0 = (2\sinh\pi)/\pi$.

b) By integrating the Fourier Series term by term from 0 to u, show that

$$a_r = \frac{-b_r}{r} \quad \text{and} \quad b_r = \frac{2(-1)^{r+1}\sinh\pi + \pi a_r}{\pi r} \quad \text{for } r \geq 1.$$

c) By substituting the expression for b_r into the expression for a_r, show that

$$a_r = \frac{2(-1)^r \sinh\pi}{\pi(r^2+1)} \quad \text{and} \quad b_r = \frac{2(-1)^{r+1} r \sinh\pi}{\pi(r^2+1)} \quad \text{for } r \geq 1.$$

(You may use the fact that the Fourier series expansion of t is

$$\sum_{r=1}^{\infty} \frac{2(-1)^{r+1}}{r} \sin rt. \quad)$$

[15 marks]

13. Calculate the Fourier series expansion of $|t|$ $(t \in [-\pi, \pi))$. By applying Parseval's theorem, show that

$$\sum_{r=1}^{\infty} \frac{1}{(2r-1)^4} = \frac{\pi^4}{96}.$$

[15 marks]