SECTION A

1. State (without proof) whether or not each of the following sequences (x_n) is i) increasing; ii) decreasing; iii) bounded above; iv) bounded below. State also the supremum, infimum, maximum, and minimum of each sequence for which they exist.

a) $x_n = (n-1)^2$ $(n \geq 0)$.

b) $x_n = n/(n+1)$ $(n \geq 0)$. [8 marks]

2. Let the continued fraction expansion of $\sqrt[3]{2} = 1.25992105\ldots$ be given by $[a_0, a_1, a_2, a_3, \ldots]$. Using your calculator, determine a_n for $0 \leq n \leq 3$ (you do not need to write anything down other than the values of each a_n). Hence calculate the first 4 convergents to $\sqrt[3]{2}$. (Recall the formulae: $p_0 = a_0$, $p_1 = a_1 a_0 + 1$, $p_n = a_n p_{n-1} + p_{n-2}$ for $n \geq 2$; $q_0 = 1$, $q_1 = a_1$, $q_n = a_n q_{n-1} + q_{n-2}$ for $n \geq 2$).

[7 marks]

3. Let $f\colon [0,1] \to [0,1]$ be a map which has a continuous derivative $f'(x)$. What does it mean for a fixed point p of f to be unstable? State how the derivative $f'(p)$ can be used to determine whether p is unstable, and sketch a spider diagram near a stable fixed point to illustrate your answer.

For each $r \in [0,4]$, let $f_r\colon [0,1] \to [0,1]$ be given by $f_r(x) = rx(1-x)$. Determine the values of r for which the fixed point $p = 0$ of f_r is unstable.

[7 marks]

4. State without proof whether each of the following sets is open, closed, both, or neither.

a) $[0,1)$, as a subset of $\mathbf{R}$.

b) $\mathbf{Q}$, as a subset of $\mathbf{R}$.

c) $\{(x,0) : x \in [0,1]\}$, as a subset of $\mathbf{R}^2$.

d) $\{(x,y) : 1 < x^2 + y^2 < 2\}$, as a subset of $\mathbf{R}^2$.

[8 marks]

5. A student spends each hour of her life either working or drinking coffee. If she is working in a given hour, she will work the next hour with probability 3/4, and drink coffee with probability 1/4. If she is drinking coffee, then she will work the next hour with probability 1/2, and drink coffee with probability 1/2.

Write down the matrix of transition probabilities which governs her behaviour. In the long term, what proportion of her life does she spend on each activity?

[7 marks]

6. Calculate the Fourier series expansion of $|\sin t|$ $(t \in [-\pi, \pi))$. [10 marks]

7. The Fourier series expansion of $|t|$ $(t \in [-\pi, \pi))$ is

$$\pi/2 + \sum_{r=1}^{\infty} \frac{2((-1)^r - 1)}{r^2 \pi} \cos rt.$$

Using Parseval's theorem, show that

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \dots = \frac{\pi^4}{96}.$$

[8 marks]

SECTION B

8. Let

$$f(x) = x + 1 - \frac{x^2}{8}.$$

Let the sequence (x_n) be defined iteratively by $x_0 = 3$ and $x_{n+1} = f(x_n)$ for each $n \geq 0$. Prove that (x_n) is an decreasing sequence which tends to $\sqrt{8}$ as $n \to \infty$. (You may use any results from the lectures without proof, but they should be clearly stated).

Show that $|x_n - \sqrt{8}| < (1/2)^n$. How large should n be in order to ensure that x_n agrees with $\sqrt{8}$ to 50 decimal places? [15 marks]

9. What is meant by a subsequence of a sequence (x_n)? State the Bolzano-Weierstrass theorem concerning the existence of convergent subsequences of a sequence (x_n) $(x_n \in \mathbf{R})$.

Which of the following sequences (x_n) has a convergent subsequence? Give reasons for your answers.

a) $x_n = (n+2)/(n+1)$.

b) $x_n = (-1)^n \sqrt{n}$.

c) $x_n = n$th digit in the decimal expansion of $\sqrt{2}$.

d) $x_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ -n & \text{if } n \text{ is odd.} \end{cases}$

e) $x_n = n/p_n$, where p_n is the nth prime number.

[15 marks]

10. State Sharkovsky's theorem (concerning the possible sets of periods of continuous maps $f: [0,1] \to [0,1]$).

Determine the Markov graphs of the two period 4 patterns $(1\,2\,3\,4)$ and $(1\,3\,2\,4)$. Show that a continuous map $f: [0,1] \to [0,1]$ with a periodic orbit of pattern $(1\,2\,3\,4)$ must have a period 3 orbit, while one with a periodic orbit of pattern $(1\,3\,2\,4)$ need not have. What other periods of orbits must there be in each of the two cases? [15 marks]

11.a) Suppose that $f: \mathbf{R} \to \mathbf{R}$ is such that f has 2 fixed points, f^2 has 8 fixed points, f^3 has 17 fixed points, and f^4 has 48 fixed points. How many periodic orbits of periods 2, 3, and 4 does f have?

b) Let $g_r: \mathbf{R} \to \mathbf{R}$ be given by $g_r(x) = r - x^2$. Determine the values of r for which g_r has a stable period 2 orbit.

[15 marks]

12. a) Calculate the Fourier series expansion of t $(t \in [-\pi, \pi))$.

b) The Fourier series expansion of t^2 $(t \in [-\pi, \pi))$ is

$$\frac{\pi^2}{3} + \sum_{r=1}^{\infty} \frac{4(-1)^r}{r^2} \cos rt.$$

Integrate this series term by term and use your result from part a) to determine the Fourier series expansion of t^3 $(t \in [-\pi, \pi))$.

c) Under what conditions is term by term differentiation of a Fourier series expansion valid?

[15 marks]

13. What does it mean for a function $f\colon \mathbf{R} \to \mathbf{R}$ to be even? Explain why an even periodic function has no sine terms in its Fourier series expansion.

Let

$$f(t) = \begin{cases} 0 & \text{if } -\pi \le t \le \pi/2 \text{ or } \pi/2 \le t < \pi \\ 1 & \text{if } -\pi/2 < t < \pi/2 \end{cases}$$

Sketch the 2π-periodic extension of $f(t)$, and show that its Fourier series expansion is given by

$$\frac{1}{2} + \frac{2}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^r}{2r+1} \cos((2r+1)t).$$

By applying the Fourier series theorem at $t = 0$, show that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots = \frac{\pi}{4}.$$

[15 marks]