SECTION A

1. In a game for two players, A and B, each player simultaneously shows 2, 3, or 4 fingers. If the sum of the number of fingers shown is prime then player A wins, otherwise player B wins. The amount won is equal to the number of fingers shown by the loser. Calculate the outcome matrix for A.

Find the security level for each of A's pure strategies.

[6 marks]

2. In a game of 'Rustic Poker', two cards *Win* and *Lose* are placed in a hat. The two players A and B stake themselves in by placing £1 each in the kitty.

A draws a card and looks at it without letting B see it. A then **either** folds, in which case B takes the kitty and the game ends **or** A raises by putting a further £3 in the kitty.

B can then **either** fold, in which case A gets the kitty **or** call by putting £3 in the kitty and drawing the remaining card. The winner is then the player with the *Win* card.

Show that A has four pure strategies and B has two pure strategies. List the pure strategies for each player.

Show that A's expected outcome on playing the strategy (in an obvious notation) $(L \to F; W \to R)$ against B's strategy (C) is £1.50 and complete A's outcome matrix.

[9 marks]

3. Define a **saddle point** of the outcome matrix for a two-player zero-sum game. Find all the saddle points of the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 1 & -1 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

[5 marks]

4. The language AB over the alphabet $\Sigma \equiv \{a, b\}$ is defined by the regular expression

$$(ab)^*a$$

(i) Give three words from AB.

(ii) Give a recursive definition of AB.

(iii) Draw a transition diagram for a finite automaton which accepts only the words of AB (reading from the right and terminated by blank).

[8 marks]

5. Let Γ be the directed graph whose adjacency matrix is as shown below:

$$
\begin{array}{c c}
 & \begin{array}{c c c c c c} a & b & c & d & e & f \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} &
\left(\begin{array}{c c c c c c}
0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 1
\end{array}\right)
\end{array}.
$$

Thus there is a directed edge from b to d but no directed edge from d to b. Determine the number of directed walks of length 2 from a to f and write them down. Which of these are directed paths? Calculate the in-degree and the out-degree at each vertex of Γ. Find a directed Euler walk in Γ, and write down, in order, the vertices passed in this walk. [9 marks]

6. Carry out a critical path analysis for the activity given below. Find the minimum time required to complete the task. Write down the critical path as a sequence of the α_i's. For each α_i not in this path, determine the float time.

Activity	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}	α_{13}
Time	5	4	3	6	8	25	2	11	17	8	22	26	6
Prerequisites			α_1	α_1	α_2	α_2	α_2	α_4	α_4	α_3	α_7	α_7	α_6
								α_5	α_5	α_8			α_9
													α_{10}
													α_{11}

[9 marks]

7. (i)   Using the method of sentence tableau, decide whether or not the following sequent is valid:

$$(r \to \neg q) \to \neg(p \to q) \ \vdash \ (p \to q) \to r.$$

(ii)   Prove that a disjunctive normal form for $\neg\big(q \to (p \wedge q)\big)$ is $q \wedge \neg p$, and deduce a disjunctive normal form for $q \to (p \wedge q)$. Hence prove that a disjunctive normal form for

$$\big(q \to (p \wedge q)\big) \wedge \big((\neg(p \wedge q)) \to p\big) \wedge \big(p \leftrightarrow q\big)$$

is $p \wedge q$.

[10 marks]

SECTION B

8. Solve the following two-player zero-sum games, giving the optimal strategy for each player and the value of the game:

(i)
$$\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.$$

(ii)
$$\begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & -2 \end{pmatrix}.$$

(iii)
$$\begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}.$$

[15 marks]

9. (i) Consider the languages X and Y over the alphabet $\Sigma \equiv \{a, b\}$ which contain the set of words of the form

$$X: \quad a^n b^m$$
$$Y: \quad a^n b^n$$

where n, m are non-zero positive integers.

Construct productions for each of these languages and give the classification of the grammar in each case. State, for each case, whether the language can be checked by a Finite State Automaton.

(ii) Construct a Turing machine which accepts only words of the form $a^n b^n$, where n is a non-zero positive integer. You may assume that the machine starts at the rightmost element of the word and that the word is terminated by a blank at the left.

[15 marks]

10. Consider the following three sentences:

(a) $p \leftrightarrow (q \leftrightarrow r)$;

(b) $\big((r \lor \neg q) \land (p \leftrightarrow p)\big) \to \neg p$;

(c) $q \land (p \lor r)$.

Prove that none of these sentences implies either of the others. Show also that precisely one of the sentences is not true under every assignment of truth values to the propositional variables making both of the other two sentences false.
[15 marks]

11. (i)   Use the method of sentence tableau to determine whether or not the following argument is valid:

$$\exists x \Big(A(x) \to \forall y \big(B(y) \to G(x,y) \big) \Big);$$

$$\forall x \Big(A(x) \land \forall y \big(C(y) \to \neg G(x,y) \big) \Big)$$

$$\vdash \ \forall x \Big(B(x) \to \neg C(x) \Big).$$

(ii)   Write the following argument symbolically and decide whether or not it is valid.

When it rains no birds fly;

when it is not cloudy some birds fly;

therefore when it is not cloudy it does not rain.

[15 marks]