SECTION A

1. In a game for two players, A and B, each player simultaneously shows 1, 2, or 3 fingers. If the difference in the number of fingers shown is 1, then the player with the larger number wins an amount equal to the number of fingers shown by the loser. If the difference in the number of fingers shown is 2, then the player with the smaller number wins an amount equal to the number of fingers shown by the loser. If both players show equal numbers, there is a draw. Calculate the outcome matrix for A.

Find the security level for each of A's pure strategies. [6 marks]

2. A simple version of the game of 'nim' is played as follows. There are two players and, at the start, two piles on the table in front of them, each containing two matches. In turn the players take any (positive) number of matches from *one* of the piles. The player taking the last match loses. Sketch a game tree.

Show that the second player has a sure win. [7 marks]

3. Define a **saddle point** of the outcome matrix for a two-player zero-sum game. Find all the saddle points of the matrix

$$\begin{pmatrix} 2 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 1 \\ -1 & 0 & -1 \end{pmatrix}.$$

[5 marks]

4. The language ABC over the alphabet $\Sigma \equiv \{a, b, c\}$ is defined by the regular expression

$$a^*bc^*$$

(a) Give a recursive definition of ABC. [3 marks]

(b) Devise productions to generate the words of ABC. [2 marks]

(c) Draw a transition diagram for a finite automaton which accepts exactly the words of ABC (reading from the right and terminated by blank). [5 marks]

5. Let Γ be the directed graph whose adjacency matrix is as shown below:

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Thus there is a directed edge from a to e but no directed edge from e to a. Determine the number of directed walks of length 2 from c to a and write them down. Which of these are directed paths? Calculate the in-degree and the out-degree at each vertex of Γ. Find a directed Euler walk in Γ, and write down, in order, the vertices passed in this walk. [9 marks]

6. Carry out a critical path analysis for the activity given below. Find the minimum time required to complete the task. Write down the critical path as a sequence of the α_i's. For each α_i not in this path, determine the float time.

Activity	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}	α_{13}
Time	3	4	12	8	10	27	3	7	18	5	11	17	8
Prerequisites			α_1	α_1	α_2	α_2	α_3	α_4	α_4	α_6	α_6	α_7	α_9
								α_5	α_5			α_8	α_{10}
													α_{12}

[9 marks]

7. (i) Using the method of sentence tableau, decide whether or not the following sequent is valid:

$$(p \to q) \to (r \vee \neg p) \;\vdash\; p \to (q \to r).$$

(ii) Prove that a disjunctive normal form for $p \leftrightarrow q$ is $(p \wedge q) \vee (\neg p \wedge \neg q)$. Hence prove that a disjunctive normal form for

$$(p \to q) \wedge \neg(p \wedge q) \wedge \neg(\neg p \leftrightarrow q)$$

is $(\neg p) \wedge \neg q$.

[10 marks]

SECTION B

8. Solve the following two-player zero-sum games, giving the optimal strategy for each player and the value of the game:

(i)

$$\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}.$$

(ii)

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 4 \end{pmatrix}.$$

Hence, or otherwise, find the value of the following game and the optimal strategy for the row player A:

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 4 \\ -1 & 0 & -1 \end{pmatrix}.$$

[15 marks]

9. (a) Construct a finite automaton which tests whether the number of 1 bits in a binary string is even. You may assume that the machine starts at the rightmost non-blank element, that the string is terminated by a blank at the left, and that it only contains the elements 0 and 1. [7 marks]

(b) The language PALINDROME over the alphabet $\Sigma \equiv \{a, b\}$ is the set of strings of even length, each string being the same whether it is read leftwards, starting from the farthest right element, or rightwards, starting from the farthest left element. You may assume that the machine starts at the rightmost non-blank element and that the string is terminated by a blank at the left. Construct a Turing machine which accepts only the words of the language PALINDROME. [8 marks]

10. Consider the following three sentences:

(a) $(p \to q) \to (q \wedge r)$;

(b) $\big((p \vee \neg q) \wedge (r \to r)\big) \to r$;

(c) $(p \vee r) \to (p \wedge q)$.

Prove that none of these sentences implies either of the others. Show also that precisely one of the sentences is never false under every assignment of truth values to the propositional variables making both of the other two sentences true.

[15 marks]

11. (i) Use the method of sentence tableau to say what you can about the following argument:

$$\exists x\Big(A(x) \to \exists y\big(B(y) \to H(x,y)\big)\Big);$$

$$\forall y\Big(B(y) \wedge \forall x\big(H(x,y) \to \neg F(x)\big)\Big)$$

$$\vdash \ \exists x\big(F(x) \to \neg A(x)\big).$$

(ii) Write the following argument symbolically and decide whether or not it is valid.

When it is cold no visitor is happy;

When the wind blows from the South some visitors are happy;

therefore when the wind blows from the South it is not cold.

[15 marks]