2MM42 Discrete Mathematics Summer 1997

Instructions to candidates

Time allowed Two Hours and a Half

You may attempt all questions.

Answers to Section A and Section B should be written in SEPARATE books and handed in SEPARATELY.

All answers to Section A and the best THREE answers to Section B will be taken into acoount.

The marks noted indicate the relative weight of the questions.

SECTION A

1. In a game for two players, A and B, each player chooses any number from 1 to 6. If the sum of the two numbers is 7 the player with the larger number wins, otherwise the player with the smaller number wins. If the two numbers are equal then there is no score. Calculate the outcome matrix for A.
[4 marks]

2. In a game of 'Rustic Poker', two cards *Win* and *Lose* are placed in a hat. The two players A and B stake themselves in by placing £1 in the kitty.

A draws a card and looks at it without letting B see it. A then **either** folds, in which case B takes the kitty and the game ends **or** A raises by putting a further £3 in the kitty.

B can then **either** fold, in which case A gets the kitty **or** raise by putting a further £3 in the kitty and drawing the remaining card. The winner is then the player with the *Win* card.

Show that A has four pure strategies and B has two pure strategies. List the pure strategies for each player.

Show that A's expected outcome on playing the strategy (in an obvious notation) $(L \to F; W \to R)$ against B's strategy (R) is £1.50 and complete A's outcome matrix.
[10 marks]

3. Define a saddle point for a two-person zero-sum game played with pure strategies, where A's outcome matrix has entry u_{ij} corresponding to strategy A_i for A and B_j for B.

Show that, if both (A_i, B_j) and (A_k, B_l) are saddle points, then $u_{ij} = u_{kl}$.
[5 marks]

4. The language AABC over the alphabet $\Sigma \equiv \{a, b, c\}$ is a set of words of length $2n$ (where n is a positive integer) of which the left n elements of a word are all a and each of the right n elements can be either b or c.

(a) List the words of AABC of length 2 and 4.

(b) Give a recursive definition of AABC.

[4 marks]

5. Explain briefly the productions allowed in the definition of type 1, type 2, and type 3 grammars.
[5 marks]

6. Let Γ be the directed graph whose adjacency matrix is as shown below:

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}.$$

Thus there is a directed edge from a to c but no directed edge from c to a. Determine the number of directed walks of length 2 from f to a and write them down. Which of these are directed paths? Calculate the in-degree and the out-degree at each vertex of Γ. Find a directed Euler walk in Γ, and write down, in order, the vertices passed in this walk. [9 marks]

7. Carry out a critical path analysis for the activity given below. Find the minimum time required to complete the task. Write down the critical path as a sequence of the α_i's. For each α_i not in this path, determine the float time.

Activity	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}	α_{13}
Time	5	18	6	10	12	5	8	6	3	7	6	2	7
Prerequisites			α_1	α_1	α_3	α_3	α_4	α_4	α_6	α_6	α_2	α_{10}	α_2
									α_7	α_7	α_8	α_{11}	α_8

[9 marks]

8. i) Using the method of sentence tableau, decide whether or not the following sequent is valid:

$$\neg p \wedge \big(\neg r \to (r \wedge \neg q)\big) \;\vdash\; (p \to \neg q) \wedge r.$$

ii) Prove that a disjunctive normal form for $\neg\big(p \leftrightarrow (p \wedge q)\big)$ is $p \wedge \neg q$, and deduce a disjunctive normal form for $p \leftrightarrow (p \wedge q)$. Hence prove that a disjunctive normal form for

$$(\neg q \to p) \wedge (\neg p \vee \neg q) \wedge \big(p \leftrightarrow (p \wedge q)\big)$$

is $\neg p \wedge q$.

[10 marks]

SECTION B

9. A two-person zero-sum game has the following outcome matrix for the row-player A:

$$\begin{pmatrix} 2 & 3 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & -1 & 5 \end{pmatrix}$$

Show that, if only the first two strategies are considered for each player, then there is a saddle point at (A_1, B_1).

Show that, if only strategies 3 and 4 are considered for each player, then A's optimal strategy is $\frac{6}{11}A_3 + \frac{5}{11}A_4$ and find B's optimal strategy.

By considering strategies of the form

$$pA_1 + (1-p)(\frac{6}{11}A_3 + \frac{5}{11}A_4)$$

find A's optimal strategy for the full game and find the value of the game. Determine also B's optimal strategy for the full game. [15 marks]

10. Construct a Turing machine to function as a parenthesis checker. Use X to denote characters other than (,), and blank. You may assume that the expression has a blank at each end but no internal blanks and that the machine starts at the rightmost nonblank. [15 marks]

11. Consider the following three sentences:

(a) $p \to (q \leftrightarrow r)$;

(b) $\neg\big(p \to (q \vee \neg r)\big)$;

(c) $\Big(\big(\neg(p \wedge \neg r)\big) \wedge (q \leftrightarrow q)\Big) \to q$.

Prove that none of these sentences implies either of the others. Show also that precisely one of the sentences is never true under every assignment of truth values to the propositional variables making either of the other two sentences true.

[15 marks]

12. (i) Use the method of sentence tableau to say what you can about the following argument:

$$\exists x\Big(F(x) \to \forall y\big(H(x,y) \to G(y)\big)\Big);$$
$$\forall x\Big(F(x) \wedge \forall y\big(B(y) \to H(x,y)\big)\Big)$$
$$\vdash \ \forall x\big(\neg G(x) \to \neg B(x)\big).$$

(ii) Write the following argument symbolically and decide whether or not it is valid.

When it rains some birds eat worms;
when the ground is hard no birds eat worms;
therefore, when it rains the ground is not hard.

[15 marks]