2MM42 Discrete Mathematics Summer 1996

Instructions to candidates

Time allowed Two Hours and a Half

You may attempt all questions.

Answers to Section A and Section B should be written in SEPARATE books and handed in SEPARATELY.

All answers to Section A and the best THREE answers to Section B will be taken into acoount.

The marks noted indicate the relative weight of the questions.

SECTION A

1. In a game for two players, A and B, each player simultaneously shows 1, 2, or 3 fingers on one hand. If both players use the same hand then the player with the larger number of fingers showing wins points equal to the difference in the number of fingers. If the two players show opposite hands then the player with the smaller number of fingers showing wins points equal to the difference. Calculate the outcome matrix for A. [4 marks]

2. A simple version of the game of nim is played as follows. There are two players and, at the start, two piles on the table in front of them, each containing two matches. In turn the players take any (positive) number of matches from *one* of the piles. The player taking the last match loses. Sketch a game tree. Show that the second player has a sure win. [6 marks]

3. Define a **saddle point** of the outcome matrix for a two-player zero-sum game. Find all the saddle points of the matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 1 & -2 & 1 & 0 & 2 \\ -1 & 0 & -1 & 1 & 1 \end{pmatrix}.$$

[6 marks]

4. In this question we consider languages over the alphabet $\Sigma \equiv \{a, b\}$. Write down a four letter word:

(a) in $a^* \vee b^*$,

(b) in a^*b^* but not in $a^* \vee b^*$,

(c) in $(a \vee b)^*$ but not in a^*b^* or $a^* \vee b^*$.

Draw a transition diagram for a finite automaton which accepts the words of a^*b^* (reading from the right and terminated by blank). Show the start state, the halt state, and the crash state. [7 marks]

5. Select any three of type 0, type 1, type 2, and type 3 languages and give a very brief definition of each, emphasising the differences between them. [5 marks]

6. Let Γ be the directed graph whose adjacency matrix is as shown below:

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Thus there is a directed edge from a to d but no directed edge from d to a. Determine the number of directed paths of length 2 from a to c and write them down. Calculate the in-degree and the out-degree at each vertex of Γ. Find a directed Euler walk in Γ, and write down, in order, the vertices passed in this walk. [9 marks]

7. Carry out a critical path analysis for the activity given below. Find the minimum time required to complete the task. Write down the critical path as a sequence of the α_i's. For each α_i not in this path, determine the float time.

Activity	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}	α_{13}
Time	4	6	4	14	6	5	8	3	16	10	6	7	2
Prerequisites				α_1	α_1	α_2	α_5	α_5	α_2	α_3	α_4	α_3	α_9
										α_6	α_7	α_6	α_{10}
										α_8		α_8	

[9 marks]

8. i) Using the method of sentence tableau, decide whether or not the following sequent is valid:

$$(p \to \neg q) \vee (q \to \neg r) \;\; \vdash \;\; p \to (\neg q \vee \neg r).$$

ii) Prove that a disjunctive normal form for $p \leftrightarrow q$ is $(p \wedge q) \vee (\neg p \wedge \neg q)$, and find a disjunctive normal form for $\neg(p \leftrightarrow q)$. Deduce that a disjunctive normal form for

$$(p \to q) \wedge (\neg p \vee \neg q) \wedge \neg(p \leftrightarrow q)$$

is $(\neg p) \wedge q$. [10 marks]

SECTION B

9. (a) Find the values of a for which the following matrix for a two-player zero-sum game has a saddle point:

$$\begin{pmatrix} 1 & a \\ 2 & -1 \end{pmatrix}.$$

(b) Solve the following two-player zero-sum game, giving the optimal strategy for each player and the value of the game:

$$\begin{pmatrix} -1 & 2 \\ 0 & -2 \end{pmatrix}.$$

(c) Find the value of the following game and the optimal strategy for the row player A:

$$\begin{pmatrix} -1 & 1 & \frac{1}{2} \\ & & \\ 1 & -1 & -\frac{1}{2} \end{pmatrix}.$$

Find the particular optimal strategy for the column player B in which the probability of B_3 is $\frac{1}{2}$.

[15 marks]

10. Consider the language BaB over the alphabet $\{a, b\}$ which consists of expressions of the form $b^n a b^n$, where $n \geq 0$.

(a) State why it is not possible to construct an FSM to accept the language BaB.

(b) Give a recursive definition of the language BaB.

(c) Write down a set of productions for the language BaB.

(d) Construct a Turing machine which accepts the language BaB.

[15 marks]

11. Consider the following three sentences:

(a) $(p \leftrightarrow q) \to r$;

(b) $\big(\neg(\neg(p \wedge r) \to \neg q)\big) \to r$;

(c) $\big((p \vee \neg q) \wedge (r \leftrightarrow r)\big) \to \neg r$.

Prove that none of these sentences implies either of the others. Show also that precisely one of these sentences is never false under every assignment of truth values to the propositional variables making either of the other two sentences false. [15 marks]

12. (i) Use the method of sentence tableau to say what you can about the following argument:

$$\exists x\Big(F(x) \to \exists y\big(H(x,y) \to G(y)\big)\Big);$$
$$\forall x\Big(F(x) \wedge \forall y\big(B(y) \to H(x,y)\big)\Big)$$
$$\vdash \; \exists x\big(\neg G(x) \to \neg B(x)\big).$$

(ii) Write the following argument symbolically and decide whether or not it is valid.

When it is hot the visitors all bathe;
When it is Winter some visitors do not bathe;
therefore when it is Winter it is not hot.

[15 marks]