2MM31

Instructions to candidates

Answer all of section A and THREE questions from section B. The total of marks available on Section A is 55.

In this paper $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ represent unit vectors parallel to the x, y and z axes respectively.

SECTION A

1. Given the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 0 & 2 \\ 0 & 4 & 0 \end{pmatrix}$$

calculate $\det \mathbf{A}$. Deduce also $\det \mathbf{A}^T$ and $\det \mathbf{A}^{-1}$.

[4 marks]

2. Find the eigenvalues and corresponding eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 10 & 8 \end{pmatrix}.$$

Write down an invertible matrix $\mathbf{T}$ such that $\mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ is a diagonal matrix.

[8 marks]

3. Given

$$\mathbf{r} = \mathbf{i} \cos \omega t + \mathbf{j} \sin \omega t,$$

where ω is a constant, calculate $\dot{\mathbf{r}}$ and $\ddot{\mathbf{r}}$, and show that

$$\ddot{\mathbf{r}} = -\omega^2 \mathbf{r}.$$

[4 marks]

4. Given the scalar field $\phi = x^3 + y^3 + z^3$, calculate $\nabla\phi$ and $\nabla\phi.\nabla\phi$.

[4 marks]

5. Explain what is meant by a conservative force $\mathbf{F}$. For the case

$$\mathbf{F} = xy\mathbf{i} + ax^2\mathbf{j} + z^3\mathbf{k},$$

where a is a constant, calculate $\nabla \wedge \mathbf{F}$, and hence find a value of a such that $\mathbf{F}$ is conservative.

[6 marks]

6. The boundaries of a flat plate are given by the parabola $y^2 = 4ax$ for $0 < x < b$ and the line $x = b$. Calculate the area of the plate.

[5 marks]

7. Find the moment of inertia of a uniform circular disc of mass M and radius a, for an axis through its centre and normal to its plane. If the disc rotates with constant angular velocity ω about this axis, give expressions in terms of M, a and ω for the kinetic energy and angular momentum associated with the motion.

[10 marks]

8. The electric field due to a point charge q at the origin is given by

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}}{r^3},$$

where $r = |\mathbf{r}|$. Calculate the flux of $\mathbf{E}$, $\int \mathbf{E}.d\mathbf{S}$, through the surface of a sphere of radius a with centre at the origin. Is the result in fact specific to spherical surfaces?

[8 marks]

9. Use the fact that $\nabla.\nabla \wedge \mathbf{G} = 0$ for any vector field $\mathbf{G}$, and the two Maxwell equations

$$\nabla.\mathbf{E} = \frac{\rho}{\epsilon_0} \quad \text{and}$$

$$\nabla \wedge \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

to show that the current density $\mathbf{J}$ and the charge density ρ are related by the equation

$$\nabla.\mathbf{J} + \frac{\partial \rho}{\partial t} = 0.$$

Discuss briefly the physical significance of this equation.

[6 marks]

SECTION B

10. Three light springs of natural length l_i and stiffness k_i where $i : 1 \cdots 3$ are attached to two masses m_1, m_2 and placed in a straight line on a smooth horizontal table as shown in the diagram.

The ends A and B are fixed. If the two masses m_1 and m_2 are displaced from equilibrium by distances x_1 and x_2 respectively, write down the equations that govern the motion of the coupled system. In the special case $m_1 = m_2 = m$ and $k_1 = k_2 = k$, show that these equations may be written in the form

$$\ddot{\mathbf{x}} = -\frac{k}{m}\mathbf{A}\mathbf{x} \quad \text{where} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \text{and} \quad \mathbf{A} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Find the natural frequencies and normal modes of the system. If at $t = 0$ we have $x_1 = a$, $\dot{x}_1 = x_2 = \dot{x}_2 = 0$, find $x_1(t)$ and $x_2(t)$.

[15 marks]

11. The motion of a particle of mass m in a central potential $V(r)$ satisfies the following equations:

$$\begin{aligned} \tfrac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V(r) &= E \\ J &= mr^2\dot{\theta} \end{aligned} \qquad (1)$$

where J and E are constants; comment briefly on their physical significance.

Show that if we define $u = 1/r$ then

$$\frac{du}{d\theta} = -\frac{m}{J}\dot{r}$$

and hence that u satisfies the equation:

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2m}{J^2}\left[E - V(\frac{1}{u})\right].$$

If $V(r) = -\alpha/r - \beta/r^2$, where $\alpha > 0$ and $\beta = J^2/(4m)$, show that

$$\left(\frac{du}{d\theta}\right)^2 + \tfrac{1}{2}u^2 - \frac{2u}{l} = \frac{2E}{\alpha l},$$

where $l = J^2/(m\alpha)$.

Verify that the orbit of the particle is described by the equation :

$$\frac{l}{r} = 2 + \sqrt{2}e\cos(\frac{\theta - \theta_0}{\sqrt{2}})$$

where θ_0 is an arbitrary constant, and $e^2 = 2 + 2El/\alpha$.

[15 marks]

12. The equation of motion for a charge q of mass m moving in an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$ is

$$m\ddot{\mathbf{r}} = q(\mathbf{E} + \dot{\mathbf{r}} \wedge \mathbf{B})$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the position of the charge at time t, and $\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt}$.

If $\mathbf{E} = E\mathbf{i}$ and $\mathbf{B} = B\mathbf{k}$ show that

$$\begin{aligned}\ddot{x} &= \frac{qE}{m} + \omega\dot{y}\\ \ddot{y} &= -\omega\dot{x}\\ \ddot{z} &= 0\end{aligned}$$

where $\omega = qB/m$.

At time $t = 0$, the charge is at rest at the origin. Show that the subsequent motion is confined to the (x, y) plane.

Show that if B is a constant then x satisfies the equation

$$\ddot{x} + \omega^2 x = \frac{qE}{m}.$$

If E is also a constant, show that for $t \geq 0$, $x(t)$ is given by

$$x = \frac{qE}{m\omega^2}\left[1 - \cos\omega t\right]$$

and find the corresponding expression for $y(t)$.

[15 marks]

13. A rigid body is rotating about a fixed point O with angular velocity $\boldsymbol{\omega}$. If we choose O as the origin, then the velocity $\mathbf{v_i}$ of the constituent particle i of mass m_i is given by $\mathbf{v}_i = \boldsymbol{\omega} \wedge \mathbf{r}_i$, where $\mathbf{r}_i$ is the position vector of the particle i. Write down an expression for the total angular momentum $\mathbf{L}$ of the motion in terms of m_i, $\mathbf{r}_i$ and $\mathbf{v_i}$, and by substituting for $\mathbf{v}_i$ show that

$$\mathbf{L} = \sum_i m_i \left[(\mathbf{r}_i . \mathbf{r}_i)\boldsymbol{\omega} - (\mathbf{r}_i . \boldsymbol{\omega})\mathbf{r}_i \right].$$

Hence obtain an expression for the inertia tensor $\mathbf{I}$ of the body. Show that if the constituent particles are replaced by a continuous mass distribution $\rho(x, y, z)$, then the components of the inertia tensor are given by, for example:

$$\mathbf{I}_{xx} = \int \rho(x, y, z)(y^2 + z^2)\, dxdydz \quad \text{and}$$

$$\mathbf{I}_{xy} = -\int \rho(x, y, z) xy\, dxdydz.$$

Show that the inertia tensor for a uniform square plate of side a and mass M in a coordinate system $Oxyz$ where O is at one corner and the x and y axes are along the edges is given by

$$\mathbf{I} = \frac{Ma^2}{3} \begin{pmatrix} 1 & -\frac{3}{4} & 0 \\ -\frac{3}{4} & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The plate is rotated about a diagonal with angular velocity ω. Write down the corresponding angular velocity vector, and show that the angular momentum of the plate is

$$\mathbf{L} = \frac{Ma^2\omega}{12\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Finally, derive the kinetic energy of rotation in terms of M, a and ω.

[15 marks]

14. Starting from the Maxwell equation

$$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

establish Faraday's law:

$$\int_C \mathbf{E}.\mathbf{dr} = -\frac{\partial}{\partial t}\Phi$$

where $\Phi = \int_S \mathbf{B}.\mathbf{dS}$ is the magnetic flux through a surface S bounded by the closed loop C. If the loop C consists of a conducting circuit, then what is the effect of $\int_C \mathbf{E}.\mathbf{dr}$ being non-zero?

A long thin straight conductor carrying a steady current I lies in the plane of a thin square loop of wire, as shown in the diagram. The nearest side of the square is parallel to and at a distance x from the long conductor, and each side of the square is of length a.

Given that the magnitude of the magnetic field at a distance r from the wire is given by

$$B = \frac{\mu_0 I}{2\pi r},$$

show that the total magnetic flux through the square is

$$\Phi = \frac{\mu_0 I a}{2\pi} \ln\left(1 + \frac{a}{x}\right).$$

If the square is moving away from the straight conductor at a constant speed $\dot{x} = v$, show that an emf $V(x)$ is induced in the wire, where

$$V(x) = \frac{\mu_0 I a^2 v}{2\pi x(a + x)}.$$

[15 marks]