2MM31 EXAM JAN 1998

Answer all of section A and THREE questions from section B. The total of marks available on Section A is 55.

In this paper $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ represent unit vectors parallel to the x, y and z axes respectively.

SECTION A

1. Find all the eigenvalues of the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 9 \\ 5 & 7 \end{pmatrix}$$

and, for each eigenvalue, find an eigenvector of $\mathbf{A}$.

Write down an invertible 2×2 matrix $\mathbf{T}$ such that $\mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ is a diagonal matrix. Verify your construction by calculating $\mathbf{T}^{-1}$ and $\mathbf{T}^{-1}\mathbf{A}\mathbf{T}$.

[10 marks]

2. Calculate the determinants of the matrices $\mathbf{A}$ and $\mathbf{B}$ where

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 2 & 1 \\ 4 & 0 & 3 \end{pmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{pmatrix} 3 & 0 & -2 \\ 2 & -1 & -2 \\ 1 & 1 & 0 \end{pmatrix}.$$

Deduce also $\det(\mathbf{A}^{-1})$.

[5 marks]

3. A rotation about the z axis through an angle θ in three dimensions is represented by the matrix

$$\mathbf{R}_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Show that $\mathbf{R}_z$ is an orthogonal matrix. Write down $\mathbf{R}_z(-\theta)$ and comment on the result.

[4 marks]

4. A particle of mass m moves on the x-axis. It is subject to a resistive force of magnitude $\lambda m v^2$ where $v(t) \equiv \frac{dx}{dt}$ is the speed of the particle at time t, and λ is a positive constant. Show that the equation of motion of the particle may be written

$$\frac{dv}{dt} + \lambda v^2 = 0.$$

Find $v(t)$ and $x(t)$ given that at $t = 0$ the particle is projected from the origin in the positive-x direction with speed v_0.

[7 marks]

5. Evaluate the double integral

$$\int\int_S y\,dx\,dy$$

where S is the region bounded by the lines $x = 1$, $x = 2$, $y = 0$ and $y = 1/x$.

[5 marks]

6. Given that in spherical polar coordinates the infinitesimal volume element is given by

$$dV = r^2 \sin\theta \, dr d\theta d\phi$$

show that the volume V of a sphere of radius a is $V = 4\pi a^3/3$.

The density of the sphere varies with r according to the formula $\rho = Ar^2$ where A is a constant. Find the mass of the sphere in terms of a and A.

[6 marks]

7. Find the moment of inertia of a uniform circular disc of mass M and radius a, for an axis through its centre and normal to its plane.

[4 marks]

8. Given that $\phi = xyz$, and $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$, calculate $\nabla\phi$, $\nabla.\mathbf{F}$ and $\nabla.(\phi\mathbf{F})$ and verify that

$$\nabla.(\phi\mathbf{F}) = (\nabla\phi).\mathbf{F} + \phi(\nabla.\mathbf{F})$$

[7 marks]

9. Maxwell's equations of electromagnetism are as follows:

$$\nabla . \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla . \mathbf{B} = 0$$

$$\nabla \wedge \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

Using the fact that for any vector field $\mathbf{A}$,

$$\nabla \wedge (\nabla \wedge \mathbf{A}) = \nabla(\nabla . \mathbf{A}) - \nabla^2 \mathbf{A},$$

show that in regions of space such that $\mathbf{J} = \rho = 0$, both $\mathbf{E}$ and $\mathbf{B}$ satisfy the three-dimensional wave equation, which for a vector field $\mathbf{A}$ is:

$$\nabla^2 \mathbf{A} = \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2},$$

for waves travelling with speed c. For $\mathbf{E}$ and $\mathbf{B}$, give an expression for c in terms of ϵ_0 and μ_0, and discuss briefly the physical significance of c.

[7 marks]

SECTION B

10. Three light springs of natural length l_i and stiffness k_i where $i : 1 \cdots 3$ are attached to two masses m_1, m_2 and placed in a straight line on a smooth horizontal table as shown in the diagram.

The ends A and B are fixed. If the two masses m_1 and m_2 are displaced from equilibrium (in the same direction) by distances x_1 and x_2 respectively, write down the equations that govern the motion of the coupled system. In the special case $m_1 = m_2 = m$ and $k_1 = k$, $k_2 = 2k$, $k_3 = 4k$ show that these equations may be written in the form

$$\ddot{\mathbf{x}} = \frac{k}{m}\mathbf{A}\mathbf{x} \quad \text{where} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \text{and} \quad \mathbf{A} = \begin{pmatrix} -3 & 2 \\ 2 & -6 \end{pmatrix}.$$

Find the natural frequencies and normal modes of the system. If at $t = 0$ we have $x_1 = a$, $\dot{x}_1 = x_2 = \dot{x}_2 = 0$, find $x_1(t)$ and $x_2(t)$.

[15 marks]

11. Let $\mathbf{A}$ be the matrix

$$\mathbf{A} = \begin{pmatrix} 5 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 7 \end{pmatrix}.$$

Find $\det \mathbf{A}$ and the eigenvalues of $\mathbf{A}$, given that one of the eigenvalues is 6. Find the corresponding NORMALISED eigenvectors.

Find a 3×3 orthogonal matrix $\mathbf{P}$ such that

$$\mathbf{P}^T \mathbf{A} \mathbf{P} = \text{ diag } (\lambda_1, \lambda_2, \lambda_3),$$

where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of $\mathbf{A}$.

(i) Show that the surface with equation

$$5x^2 + 6y^2 + 7z^2 - 4xy - 4yz = 36$$

is an ellipsoid, and give the lengths of its semi-major and semi-minor axes and the directions of its principal axes.

(ii) The rotational kinetic energy of a certain rigid body of mass M about its centre of mass, G, is given with respect to Cartesian coordinates $Gxyz$ at G by the formula

$$T_{\text{rot}} = \frac{1}{2} M a^2 \boldsymbol{\omega}^T \mathbf{A} \boldsymbol{\omega},$$

where $\boldsymbol{\omega}$ is the angular velocity vector relative to these axes, and a is a constant length. Write down the moments of inertia about each of the principal axes through G.

[15 marks]

12. A particle of mass m moves in a plane. Starting from the equations

$$\hat{\mathbf{r}} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$$
$$\hat{\boldsymbol{\theta}} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j},$$

where $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are unit radial and transverse vectors respectively, show that the velocity $\dot{\mathbf{r}}$ of the particle is given by

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}.$$

Hence show that if the particle is subject to a central potential $V(r)$ then its total energy is given by

$$E = \tfrac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V(r)$$

where $J = mr^2\dot{\theta}$. What is the physical significance of J? Starting from the equation $\mathbf{J} = m\mathbf{r} \wedge \dot{\mathbf{r}}$, or otherwise, show that J is a constant.

Show also that

$$\frac{dr}{d\theta} = \frac{mr^2}{J}\dot{r}$$

and hence that:

$$\left(\frac{dr}{d\theta}\right)^2 + r^2 = \frac{2mr^4}{J^2}\left[E - V\right].$$

Find an expression (in terms of E, J, m, c and r) for the potential $V(r)$ that will produce a spiral orbit of the form $r = c\theta^2$, where c is a constant.

[15 marks]

13. A rigid body is rotating about a fixed point O with instantaneous angular velocity $\boldsymbol{\omega}$. If we choose O as the origin, then show that any vector $\mathbf{A}$ satisfies the equation

$$\frac{d\mathbf{A}}{dt} = \dot{\mathbf{A}} + \boldsymbol{\omega} \wedge \mathbf{A}.$$

where $\frac{d\mathbf{A}}{dt}$ and $\dot{\mathbf{A}}$ represent the rate of change of $\mathbf{A}$ as measured by an inertial coordinate system and one rotating with the body respectively.

Hence choosing $\mathbf{A} = \mathbf{L}$ where $\mathbf{L}$ is the angular momentum of the body, show that if it is rotating freely (i.e. in the absence of torques) then $\boldsymbol{\omega}$ satisfies the following equations:

$$\begin{aligned} I_1\dot{\omega}_1 &= \omega_2\omega_3(I_2 - I_3) \\ I_2\dot{\omega}_2 &= \omega_3\omega_1(I_3 - I_1) \\ I_3\dot{\omega}_3 &= \omega_1\omega_2(I_1 - I_2) \end{aligned}$$

where the components of $\boldsymbol{\omega}$ are taken along the directions of the principal axes and $I_1 \cdots I_3$ are the corresponding principal moments of inertia.

Hence show that a body may rotate freely about a principal axis, with constant angular velocity.

For the special case $I_1 = I_2 = I$, show that a possible solution to the equations is

$$\boldsymbol{\omega} = (A\cos\Omega t, A\sin\Omega t, B)$$

where A and B are arbitrary constants, and Ω is a constant you should determine in terms of I, I_3 and B.

[15 marks]

14. The vector field $\mathbf{F}$ is given by

$$\mathbf{F} = xy^2\mathbf{i} + a(x^2y + yz^2)\mathbf{j} + y^2z\mathbf{k},$$

where a is a constant.

Evaluate the line integrals

$$L_1 = \int_{C_1} \mathbf{F}.\mathbf{dr}$$

and

$$L_2 = \int_{C_2} \mathbf{F}.\mathbf{dr}$$

where C_1 is the curve in the xy plane, $y = x^2$, from $(0,0,0)$ to $(1,1,0)$, and C_2 is the straight line $y = x$ joining the same two points.

Find a value of a such that $L_1 = L_2$.

Calculate curl $\mathbf{F}$ and comment on the above result for a in the light of Stoke's theorem for vector integrals.

For general a, verify Stoke's theorem by calculating $\int_S \operatorname{curl} \mathbf{F}.d\mathbf{S}$, where S is the region of the xy plane bounded by C_1 and C_2.

[15 marks]